

DEFORMATION QUANTIZATION OF A_∞ -MORITA EQUIVALENCES

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We show that Deformation Quantization of quadratic Poisson structures preserves the A_∞ -Morita equivalence of a given pair of Koszul dual A_∞ -algebras.

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1. INTRODUCTION

In this paper we consider a finite dimensional vector space X over a field \mathbb{K} of characteristic 0 and the associative algebras with zero differentials $A = S(X^*)$ resp. $B = \wedge(X)$ i.e. the symmetric algebra over X^* , resp. the exterior algebra over X . For simplicity we choose $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In [3] it is shown that it is possible to endow $K = \mathbb{K}$ with an A_∞ - A - B -bimodule given by a codifferential d_K whose Taylor components are defined by certain perturbative expansions in Feynman diagrams. The expansions are written by considering configuration spaces of points on the complex upper half plane and differential 1-forms called the 4-colors propagators. This construction and those in [5],[6] are the first partial example of multi-brane generalization of the results by M. Kontsevich on Deformation Quantization of Poisson manifolds; see [13]. In [3] it is shown that the A_∞ - A - B -bimodule (K, d_K) is s.t. the classical Koszul duality between A and B holds, i.e. there exists isomorphisms

$$A \simeq \text{Ext}_B(\mathbb{K}, \mathbb{K}), \quad B \simeq \text{Ext}_A(\mathbb{K}, \mathbb{K})^{op},$$

of algebras: as left A_∞ - A -module and right A_∞ - B -module K is in fact the classical augmentation module.

Our first goal is to prove an A_∞ -derived Morita equivalence for the pair (A, B) explicitly, i.e. the equivalence of certain triangulated subcategories of the derived categories of strictly unital A_∞ -right-modules over A and B by using the A_∞ -bimodule (K, d_K) : A and B are just associative algebras with zero differential but we consider categories of A_∞ -modules over them.

It is natural to introduce a bigrading on the triple (A, K, B) ; the first grading is cohomological; the second grading is called internal; consequently we consider only bigraded A_∞ -structures, i.e. bigraded A_∞ modules, bimodules, morphisms between them etc. By definition, the internal grading is preserved by the A_∞ -structures and morphisms between them.

The A_∞ -Morita equivalence for the pair (A, B) has been already proved in [27], where a more general result is shown. In [27], (see prop. 1.14, 3.1. and thm. 5.7, 5.8 *loc. cit.*) the authors prove the aforementioned equivalence by “returning” to the differential bigraded level by considering the derived categories of differential bigraded modules over the enveloping algebras UA , resp. UB of A resp. B . The enveloping algebra UA' of any bigraded A_∞ -algebra A' is a differential bigraded algebra. It is introduced in [27] as the theory of differential bigraded algebras is, in general, simpler than the theory of bigraded A_∞ -algebras. Such an approach has the advantage of using the already

well-known results on the enveloping algebras and (bar) resolutions of differential bigraded algebras. On the other way, using this approach one introduces the iterated use of the Koszul dual functor $E(\cdot)$, which associates to any augmented A_∞ -algebra A' its A_∞ -Koszul dual $E(A') = \text{Hom}(UA', \mathbb{K})$. Moreover the enveloping algebra UA' is a rather “big” bigraded object, as by definition it is the cobar construction of the bar construction over A' .

Our approach is alternative to the one presented in [27]; we use the A_∞ -bimodule K to prove the Morita equivalence at the A_∞ -level, without using the enveloping algebras UA, UB and returning to the differential bigraded level.

The key observation in our construction is that the left derived derived actions ([10], [3])

$$L_A : A \rightarrow \underline{\text{End}}_B(K), \quad R_B : B \rightarrow \underline{\text{End}}_A(K)^{op},$$

are quasi-isomorphisms of strictly unital A_∞ - A - A -bimodules and strictly unital A_∞ - B - B -bimodules; this is done in subsection 5.0.9. We use this fact to prove the equivalences of categories before and after deformation quantization.

The pair of functors inducing the equivalence is studied in subsection 6.0.16. We define them by using the tensor products $\bullet \otimes_A \bullet$, $\bullet \otimes_B \bullet$ of A_∞ -modules described in subsection 4.0.6. The main advantage of such “pure” A_∞ -approach, aside from the explicit use of the bimodule K , is represented by the possibility of quantizing the equivalences: this is the content of section 8. Let $\hbar\pi$ be an \hbar -formal quadratic Maurer-Cartan-element of cohomological degree 1 in $T_{poly}(X)[[\hbar]]$, the ring of formal power series in \hbar with coefficients in $T_{poly}^\bullet(X) = S(X^*) \otimes \wedge^{\bullet+1}(X)$. $T_{poly}(X)[[\hbar]]$ is a differential graded Lie algebra with zero differential and graded Lie bracket $[\cdot, \cdot]_\hbar$ obtained by extending $\mathbb{K}[[\hbar]]$ -linearly the Schouten-Nijenhuis bracket $[\cdot, \cdot]$ on $T_{poly}(X)$. With such a choice of Poisson bivector the internal grading on the triple on (A, K, B) is preserved; i.e. using the “2-branes Formality theorem” contained in [3] it follows that the quantizations A_\hbar , resp. B_\hbar of A , resp. B are associative bigraded algebras with zero differentials. The quantized bimodule $K_\hbar = (K[[\hbar]], d_{K_\hbar})$ satisfies the quantized version of the Keller condition, and it is a left A_\hbar -module and a right B_\hbar -module with zero differential. Moreover, it is possible to quantize straightforwardly the bar resolutions $A \otimes_A K$, $K \otimes_B B$ and the A_∞ -bimodules introduced in section 6.

In this “deformed” or quantized setting we introduce the categories $\mathbf{Mod}_{tf}^\infty(A_\hbar)$, resp. $\mathbf{Mod}_{tf}^\infty(B_\hbar)$ of strictly unital topologically free right A_∞ - A_\hbar -modules (resp. A_∞ - B_\hbar -modules). An object N_\hbar in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ is a collection $\{N_j^i[[\hbar]]\}_{i,j \in \mathbb{Z}}$ of topologically free $\mathbb{K}[[\hbar]]$ -modules, endowed with a topological A_∞ - A_\hbar -module structure, i.e. a quantized codifferential $d_{M_\hbar} = \sum_{i \geq 0} d_{M_\hbar}^{(i)} \hbar^i$ s.t. $d_{M_\hbar} \circ d_{M_\hbar} = 0$.

$\mathbf{Mod}_{tf}^\infty(A_\hbar)$ and $\mathbf{Mod}_{tf}^\infty(B_\hbar)$ are additive categories but they are not closed under taking cohomology. Quasi-isomorphisms in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ (and $\mathbf{Mod}_{tf}^\infty(B_\hbar)$) are then defined by considering the bigger abelian category $\mathbf{Mod}_{bg}(\mathbb{K}[[\hbar]])$ of all bigraded $\mathbb{K}[[\hbar]]$ -modules.

We define the homotopy categories $\mathcal{H}_\infty^{tf}(A_\hbar)$, $\mathcal{H}_\infty^{tf}(B_\hbar)$; these are naturally triangulated categories. To prove this we define topological cones and cylinders of topological A_∞ -morphisms; all details are contained in subsection 8.0.33. Our approach is quite “down-to-earth”: we adapt the definitions and results in [20] to our topological A_∞ -setting.

We finish by introducing “the derived categories” $\mathbf{D}_{tf}^\infty(A_\hbar)$ resp. $\mathbf{D}_{tf}^\infty(B_\hbar)$ as the localization at quasi-isomorphisms of $\mathcal{H}_\infty^{tf}(A_\hbar)$ resp. $\mathcal{H}_\infty^{tf}(B_\hbar)$: they are canonically endowed with a triangulated category structure induced by the one on the corresponding homotopy category.

With $\mathbf{triang}_{A_\hbar}^\infty(M_\hbar)$, $\mathbf{triang}_{B_\hbar}^\infty(N_\hbar)$ we denote the full triangulated subcategories in $\mathbf{D}_{tf}^\infty(A_\hbar)$ resp. $\mathbf{D}_{tf}^\infty(B_\hbar)$ generated by $\{M_\hbar[i]\langle j \rangle\}_{i,j \in \mathbb{Z}}$ and similarly for N_\hbar , where $[\cdot]$, resp. $\langle \cdot \rangle$ are the shifts w.r.t. the cohomological resp. internal grading. With $\mathbf{thick}_{A_\hbar}^\infty(M_\hbar)$ and $\mathbf{thick}_{B_\hbar}^\infty(N_\hbar)$ we denote their thickenings. We recall their definitions in Appendix C. Let $\tilde{\otimes}$ be the completed tensor product of bigraded $\mathbb{K}[[\hbar]]$ -modules w.r.t. the \hbar -adic topology. The completed tensor products $\bullet \tilde{\otimes}_{A_\hbar} \bullet$, $\bullet \tilde{\otimes}_{B_\hbar} \bullet$ of topological A_∞ -modules are defined accordingly.

The main result of these notes is then

Theorem 1. *Let X be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$, or \mathbb{C} and (A, K, B) be the triple of bigraded A_∞ -structures with $A = S(X^*)$, $B = \wedge(X)$ and $K = \mathbb{K}$ endowed with the A_∞ - A - B -bimodule structure given in [3]. By $\pi_\hbar \in (T_{poly}(X)[[\hbar]], 0, [\cdot, \cdot]_\hbar)$ we denote an \hbar -formal quadratic Poisson bivector on X and by $(A_\hbar, K_\hbar, B_\hbar)$ the Deformation Quantization of (A, K, B) w.r.t. π_\hbar . The triangulated functor*

$$\mathcal{F}_\hbar : \mathbf{D}_{tf}^\infty(A_\hbar) \rightarrow \mathbf{D}_{tf}^\infty(B_\hbar), \quad \mathcal{F}_\hbar(\bullet) = \bullet \tilde{\otimes}_{A_\hbar} K_\hbar$$

induces the equivalence of triangulated categories

$$\mathbf{triang}_{A_\hbar}^\infty(A_\hbar) \simeq \mathbf{triang}_{B_\hbar}^\infty(K_\hbar), \quad \mathbf{thick}_{A_\hbar}^\infty(A_\hbar) \simeq \mathbf{thick}_{B_\hbar}^\infty(K_\hbar).$$

Let $(\tilde{K}, d_{\tilde{K}})$ be the A_∞ - B - A -bimodule with $\tilde{K} = K$ and $d_{\tilde{K}}$ obtained from d_K exchanging A and B ; the triangulated functor

$$\mathcal{F}_\hbar'' : \mathbf{D}_{tf}^\infty(B_\hbar) \rightarrow \mathbf{D}_{tf}^\infty(A_\hbar), \quad \mathcal{F}_\hbar''(\bullet) = \bullet \tilde{\otimes}_{B_\hbar} \tilde{K}_\hbar$$

induces the equivalence of triangulated categories

$$\mathbf{triang}_{A_h}^\infty(\tilde{K}_h) \simeq \mathbf{triang}_{B_h}^\infty(B_h), \quad \mathbf{thick}_{A_h}^\infty(\tilde{K}_h) \simeq \mathbf{thick}_{B_h}^\infty(B_h).$$

In other words, Deformation Quantization of \hbar -formal quadratic Poisson bivectors preserves the A_∞ -Morita equivalence of the Koszul dual A_∞ -algebras A and B .

In Appendix A we show the proof of prop. 6, while in Appendix B-C we prove thm. 7 and thm. 9 in some detail. Such proof are conceptually quite easy; using the very definition of the triangulated subcategories $\mathbf{triang}_{A_h}^\infty(A_h) \dots \mathbf{thick}_{B_h}^\infty(K_h)$ we just need to check the commutativity of diagrams in which the quasi-isomorphisms of A_∞ -bimodules of section 6 appear. Moreover, the proof of thm. 9 is analogous to the one of thm. 7, with mild changes.

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3. NOTATION AND CONVENTIONS

Let \mathbb{K} be a field of characteristic 0. Throughout this work we fix $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $\mathbf{bG}_\mathbb{K}$ be the category of \mathbb{Z} -bigraded vector spaces, i.e. collections $\{M_j^i\}_{i,j \in \mathbb{Z}}$ of vector spaces over \mathbb{K} . The upper grading is also called the “cohomological grading”. The lower index denotes the “internal grading”. The space of morphisms $\mathrm{Hom}_{\mathbf{bG}_\mathbb{K}}(M, N)$ is the \mathbb{Z} -bigraded vector space with $(r, s)^{th}$ component

$$\mathrm{Hom}_{\mathbf{bG}_\mathbb{K}}^{r,s}(M, N) = \prod_{n, m \in \mathbb{Z}} \mathrm{Hom}_\mathbb{K}(M_m^n, N_{m+s}^{n+r}),$$

for every $r, s \in \mathbb{Z}^2$.

Any $f \in \mathrm{Hom}_{\mathbf{bG}_\mathbb{K}}^{r,s}(M, N)$ is said to be a bigraded morphism of bidegree (r, s) . The identity morphisms in $\mathbf{bG}_\mathbb{K}$ are denoted simply by 1. For any object M in $\mathbf{bG}_\mathbb{K}$, we denote by $M[n]$ the object in $\mathbf{bG}_\mathbb{K}$ such that $(M[n])_j^i := M_j^{i+n}$; the degree -1 isomorphism $s : M \rightarrow M[1]$, $s(m) := m$ is called the suspension map; its inverse of degree 1 $s^{-1} : M[1] \rightarrow M$ is the desuspension. Both are endofunctors of $\mathbf{bG}_\mathbb{K}$, with $(s^{-1})^{\otimes i} \circ s^{\otimes i} = (-1)^{\frac{i(i-1)}{2}} 1$. We use the short notation sm for $s(m) \in M[1]$. The cohomological degree of bihomogeneous elements of M is denoted by $|\cdot|$; in particular $|sm| = |m| - 1$, for every $sm \in M[1]$.

Similarly, the object $M\langle j \rangle$ in $\mathbf{bG}_\mathbb{K}$ is s.t. $M\langle j \rangle_m^n := M_{m+j}^n$, for any $j \in \mathbb{Z}$. It follows that $\mathrm{Hom}_{\mathbf{bG}_\mathbb{K}}^{r,s}(M, N) = \mathrm{Hom}_{\mathbf{bG}_\mathbb{K}}^{0,0}(M, N[r]\langle s \rangle)$.

The tensor product $M \otimes N$ of any two objects in $\mathbf{bG}_\mathbb{K}$ is the object in $\mathbf{bG}_\mathbb{K}$ with bihomogeneous components

$$(M \otimes N)_m^n = \bigoplus_{\substack{p+q=n \\ r+s=m}} M_r^p \otimes N_s^q,$$

for every $n, m \in \mathbb{Z}$ with $\otimes = \otimes_\mathbb{K}$. Throughout this work we will use the shorthand conventions $m_1, \dots, m_n \stackrel{!}{=} m_1 \otimes \dots \otimes m_n$, and $(m_1 | \dots | m_n) \stackrel{!}{=} s(m_1) \otimes \dots \otimes s(m_n)$, for any $m_1, \dots, m_n \in M \in \mathbf{bG}_\mathbb{K}$. So, in particular, $(m_1, m_2 | m_3) = m_1 \otimes s(m_2) \otimes s(m_3)$ and $(m_1 | m_2, m_3) = s(m_1) \otimes s(m_2) \otimes m_3$. In what follows we assume that the Koszul sign rule holds.

4. A_∞ -STRUCTURES

In this section we introduce A_∞ -structures from a purely algebraic point of view. We recall the concept of A_∞ -algebra, A_∞ -module, A_∞ -bimodule and their morphisms. We focus our attention on unital A_∞ -structures, augmented A_∞ -algebras. The tensor product of A_∞ -modules is also considered; it contains the bar resolution of a module over a given unital algebra as special case. A_∞ -algebras have been introduced by Stasheff [25] in the sixties in algebraic topology; in the nineties they have been further popularized by Kontsevich’s [14] in his Homological Mirror Symmetry conjecture. The material here presented is standard; we refer to [12, 9, 19, 26] for all details, in particular the definitions of coalgebras, coderivations, comodules etc... For the interested reader, we just note that such definitions can be deduced by taking the “limit” $\hbar = 0$ in the formulæ appearing in section 8. Tensoring of A_∞ -bimodules has been introduced explicitly in [18], extending the case of right A_∞ -modules contained in [12]. In what follows we will consider only bigraded A_∞ -structures; the rule of thumb is that the maps defining the A_∞ -structures themselves preserve the internal grading. In this sense, there is not substantial difference between the graded and bigraded case.

4.0.1. *A_∞-algebras.* Let A be an object of $\mathbf{bG}_{\mathbb{K}}$. The coassociative counital tensor coalgebra on A is the triple

$$\mathcal{B}(A) := (\mathrm{T}^c(A[1]), \Delta, \epsilon),$$

where $\mathrm{T}^c(A[1]) = \bigoplus_{k \geq 0} A[1]^{\otimes k}$, $\Delta : \mathrm{T}^c(A[1]) \rightarrow \mathrm{T}^c(A[1]) \otimes \mathrm{T}^c(A[1])$ is the coassociative coproduct $\Delta(a_1 | \dots | a_n) = 1 \otimes (a_1 | \dots | a_n) + (a_1 | \dots | a_n) \otimes 1 + \sum_{n'=1}^{n-1} (a_1 | \dots | a_{n'}) \otimes (a_{n'+1} | \dots | a_n)$ and the counit ϵ denotes the projection onto \mathbb{K} ; by definition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta = 1$.

Definition 1 (J. Stasheff, [25]). *An A_{∞} -algebra is a pair (A, d_A) , where A is an object of $\mathbf{bG}_{\mathbb{K}}$ and d_A is a bidegree $(1, 0)$ coderivation on $\mathcal{B}(A)$ s.t.*

$$d_A \circ d_A = 0.$$

By the lifting property of coderivations on $\mathcal{B}(A)$, such d_A is uniquely determined by its Taylor components, i.e. the family of morphisms $\bar{d}_A^n := \mathrm{pr}_{A[1]} \circ d_A|_{A[1]^{\otimes n}}$, $n \geq 0$, denoting by $\mathrm{pr}_{A[1]}$ the projection $\mathrm{pr}_{A[1]} : \mathrm{T}^c(A[1]) \rightarrow A[1]$.

Then $d_A \circ d_A = 0$ is equivalent to

$$(1) \quad \sum_{s_1=0}^k \sum_{j=1}^{k-s_1+1} (-1)^{\epsilon} \bar{d}_A^{k-s_1+1}(a_1 | \dots | a_{j-1}, \bar{d}_A^{s_1}(a_j | \dots | a_{s_1+j-1}) | a_{s_1+j} | \dots | a_k) = 0,$$

for every $k \geq 0$ and $(a_1, \dots, a_k) \in \mathcal{B}(A)$. The Koszul sign is simply $\epsilon = \sum_{i=1}^{j-1} (|a_i| - 1)$. Equivalently, we can consider the bidegree $(2 - n, 0)$ maps m_n defined through

$$(2) \quad \begin{aligned} \bar{d}_A^0 &= -s \circ m_0, \\ \bar{d}_A^n &= -s \circ m_n \circ (s^{-1})^{\otimes n}, \quad n \geq 1, \end{aligned}$$

An A_{∞} -algebra (A, d_A) is said to be flat if $\bar{d}_A^0 = 0$. In this case m_1 is a differential and m_2 is associative up to homotopy. It reduces to an associative product on the cohomology $H(A)$ with respect to m_1 . If a flat A_{∞} -algebra is s.t. $m_3 = m_4 = \dots = 0$, then it is a differential bigraded algebra. If (A, d_A) is not flat, then it is called curved, with curvature \bar{d}_A^0 (or $\bar{d}_A^0(1)$; we use both notations). In presence of non trivial curvature, \bar{d}_A^1 is not a differential. Any graded associative algebra A s.t. $\bar{d}_A^0(1)$ is a degree 2 element in the center of A is a curved A_{∞} -algebra. Curvature appears naturally in Deformation Quantization: see for example [2]. Curved A_{∞} -algebras are also related to models in theoretical physics [4]. With a little abuse of notation we introduce the following

Definition 2. *Let (A, d_A) and (B, d_B) be A_{∞} -algebras. A morphism $F : A \rightarrow B$ of A_{∞} -algebras is a morphism $F \in \mathrm{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{0,0}(\mathcal{B}(A), \mathcal{B}(B))$ of coassociative counital coalgebras s.t.*

$$F \circ d_A = d_B \circ F.$$

$F : \mathrm{T}^c(A[1]) \rightarrow \mathrm{T}^c(B[1])$ is uniquely determined by the family of morphisms $F_n : A[1]^{\otimes n} \rightarrow B[1]$ s.t. $\mathrm{pr}_{B[1]} \circ F|_{A[1]^{\otimes n}} = F_n$ and $F(1) = 1$. The morphisms F_n are called the Taylor components of F . $F \circ d_A = d_B \circ F$ is equivalent to a tower of quadratic relations involving the Taylor components F_{\bullet} , \bar{d}_A^{\bullet} and \bar{d}_B^{\bullet} of F , A and B , respectively. If (A, d_A) , resp. (B, d_B) , are curved A_{∞} -algebras with curvature \bar{d}_A^0 , resp. \bar{d}_B^0 , then, by definition of F : $F_1(\bar{d}_A^0(1)) = \bar{d}_B^0(1)$.

It is useful to introduce the degree $1 - n$ desuspended morphisms $f_n : A^{\otimes n} \rightarrow B$ in $\mathbf{bG}_{\mathbb{K}}$, through

$$(3) \quad F_n = s \circ f_n \circ (s^{-1})^{\otimes n},$$

for every $n \geq 0$. A morphism $F : A \rightarrow B$ of A_{∞} -algebras is said to be strict if $F_n = 0$ for $n \geq 2$. If A and B are flat, F is a quasi-isomorphism if F_1 is a quasi-isomorphism in $\mathbf{bG}_{\mathbb{K}}$.

4.0.2. *Units and augmentations in flat A_{∞} -algebras.* Let (A, d_A) be an A_{∞} -algebra; the maps m_n , $n \geq 0$ and f_m , $m \geq 1$, have been defined in (2), (3).

Definition 3. *An A_{∞} -algebra (A, d_A) is said to be strictly unital if it contains an element $1_A \in A_0^0$ s.t.*

$$m_2(a, 1_A) = m_2(1_A, a) = a,$$

for any $a \in A$ and $m_n(a_1, \dots, a_n) = 0$ for $n \geq 3$ if $a_i = 1$ for some $i = 1, \dots, n$.

We note that, if A is strictly unital, then $\bar{d}_A^1(s1_A) = 0$, also in presence of curvature on A .

A morphism $F : A_1 \rightarrow A_2$ of strictly unital A_{∞} -algebras is said to be strictly unital if

$$f_1(1_{A_1}) = 1_{A_2},$$

and $f_m(a_1, \dots, a_m) = 0$ for $m \geq 2$ if $a_i = 1_{A_1}$ for some $i = 1, \dots, m$. In particular, it follows that $\bar{d}_B^1(F_1(1_A)) = 0$.

Lemma 1. *Any strictly unital flat A_{∞} -algebra A with unit 1_A comes equipped with a strict strictly unital morphism $\eta : K \rightarrow A$, sending the unity 1 of the ground field \mathbb{K} to 1_A .*

This allows us to introduce the following

Definition 4. A strictly unital flat A_∞ -algebra (A, d_A) with unit 1_A is augmented if there exists a strictly unital A_∞ -algebra morphism $\epsilon : A \rightarrow K$, s.t. $\epsilon \circ \eta = 1$.¹

We note that the morphism $\epsilon \circ \eta$ is strict as ϵ is strictly unital. If A is an augmented A_∞ -algebra with augmentation ϵ , then we call $\ker \epsilon_1$ the augmentation ideal of A .

4.0.3. A_∞ -modules and A_∞ -bimodules. In this subsection (A, d_A) and (B, d_B) are A_∞ -algebras.

Definition 5. A left A_∞ - A -module is pair (M, d_M) , where M is an object in \mathbf{bG}_K and $d_M \in \text{Hom}_{\mathbf{bG}_K}^{1,0}(\mathcal{L}(M), \mathcal{L}(M))$ is a codifferential on $\mathcal{L}(M) := T(A[1]) \otimes M[1]$ s.t.

$$d_M \circ d_M = 0.$$

As in the case of morphisms and coderivations on the tensor coalgebra $T^c(V)$, the codifferential d_M is uniquely determined by its Taylor components $\bar{d}_M^s : A[1]^{\otimes s} \otimes M[1] \rightarrow M[1]$, $s \geq 0$, via

$$d_M^k = \sum_{s_1=0}^k \sum_{j=1}^{k-s_1+1} 1^{\otimes j-1} \otimes \bar{d}_A^{s_1} \otimes 1^{\otimes k-s_1-j+1} + \sum_{s=0}^k 1^{\otimes k-s} \otimes \bar{d}_M^s,$$

where the $\bar{d}_A^{s_1}$ denote the Taylor components of the coderivation d_A defining the A_∞ -algebra structure on A .

Let (M, d_M) be a left A_∞ - A -module. $d_M \circ d_M = 0$ is equivalent to

$$(4) \quad \sum_{s_1=0}^k \sum_{j=1}^{k-s_1+1} (-1)^{\epsilon_1} \bar{d}_M^{k-s_1+1}(a_1 | \dots | a_{j-1}, \bar{d}_A^{s_1}(a_j | \dots | a_{s_1+j-1}) | a_{s_1+j} | \dots | a_k | m) + \sum_{s_2=0}^k (-1)^{\epsilon_2} \bar{d}_M^{k-s_2}(a_1 | \dots | a_{k-s_2}, \bar{d}_M^{s_2}(a_{k-s_2+1} | \dots | a_k | m)) = 0,$$

with $\epsilon_1 = \sum_{i=1}^{j-1} (|a_i| - 1)$, $\epsilon_2 = \sum_{i=1}^{k-s_2} (|a_i| - 1)$.

Remark 2. With obvious changes it is possible to define right A_∞ - A -modules on the right $\mathcal{B}(A)$ -counital comodule $\mathcal{R}(M) = M[1] \otimes T(A[1])$.

If A is curved then $\bar{d}_M^1(d_A^0(1), sm) + \bar{d}_M^0(d_M^0(sm)) = 0$, i.e. in presence of non trivial curvature $d_A^0(1)$, \bar{d}_M^0 is not a differential on $M[1]$.

Definition 6. A morphism $F : M \rightarrow N$ of left A_∞ -modules (M, d_M) , (N, d_N) is a morphism $F \in \text{Hom}_{\mathbf{bG}_K}^{0,0}(\mathcal{L}(M), \mathcal{L}(N))$ of left- $\mathcal{B}(A)$ -counital-comodules s.t.

$$(5) \quad F \circ d_M = d_N \circ F.$$

Any morphism $F : M \rightarrow N$ of left A_∞ -modules is uniquely determined by its Taylor components $F_n : A[1]^{\otimes n} \otimes M[1] \rightarrow N[1]$. Eq. (5) is equivalent to a tower of quadratic relations involving the Taylor components F_n , \bar{d}_M^\bullet , \bar{d}_N^\bullet and \bar{d}_A^\bullet ; if A is curved then

$$F_0(\bar{d}_M^0(sm)) + F_1(\bar{d}_A^0(1), sm) = \bar{d}_N^0(F_0(sm));$$

i.e. in presence of non trivial curvature $d_A^0(1)$, $F_0 : M[1] \rightarrow N[1]$ does not commute with d_M^0 and d_N^0 (which are not differentials).

Definition 7. A morphism $F : M \rightarrow N$ of left- A_∞ - A -modules is said to be strict if $F_n = 0$ for $n \geq 1$. If A is flat, F is a quasi-isomorphism if F_0 is a quasi-isomorphism.

Definition 8. An A_∞ - A - B -bimodule is a pair (M, d_M) , where M is an object in \mathbf{bG}_K and $d_M \in \text{Hom}_{\mathbf{bG}_K}^{1,0}(\mathcal{B}(M), \mathcal{B}(M))$ is a codifferential on $\mathcal{B}(M) = T(A[1]) \otimes M[1] \otimes T(B[1])$ s.t.

$$d_M \circ d_M = 0.$$

¹For any A_∞ -algebra B , the identity morphism $1 : B \rightarrow B$ is the strict A_∞ -morphism with non trivial Taylor component $\bar{1}^1(b) = b$, for every $b \in B$.

Once again, it is possible to show that the codifferential d_M is uniquely determined by the Taylor components $\bar{d}_M^{k,l} := A[1]^{\otimes k} \otimes M[1] \otimes B[1]^{\otimes l} \rightarrow M[1]$, $k, l \geq 0$, with

$$\begin{aligned} d_M^{k,l} &= \sum_{s_1=0}^k \sum_{j=1}^{k-s_1+1} 1^{\otimes j-1} \otimes \bar{d}_A^{s_1} \otimes 1^{\otimes k-j-s_1+1+l+1} + \sum_{s_2=0}^l \sum_{j=1}^{l-s_2+1} 1^{\otimes k+1} \otimes 1^{\otimes j-1} \otimes \bar{d}_B^{s_2} \otimes 1^{\otimes l-j-s_2+1} + \\ &\quad \sum_{s_3=0}^k \sum_{s_4=0}^l 1^{\otimes k-s_3} \otimes \bar{d}_M^{s_3,s_4} \otimes 1^{\otimes l-s_4}. \end{aligned}$$

Then $d_M \circ d_M = 0$ is equivalent to a tower of quadratic relations similar to (4), with due differences. In presence of non trivial curvatures on A and/or B , then $\bar{d}_M^{0,0}$ is not a differential on $M[1]$.

Lemma 2 ([18]). *Let (A, d_A) , (B, d_B) be A_∞ -algebras and (M, d_M) be an A_∞ - A - B -bimodule.*

- *If B is flat, then the family $\bar{d}_M^{k,0} : A[1]^{\otimes k} \otimes M[1] \rightarrow M[1]$ defines a left- A_∞ - A -module structure on M .*
- *If A is flat, then the family $\bar{d}_M^{0,l} : M[1] \otimes B[1]^{\otimes l} \rightarrow M[1]$, $l \geq 0$, defines a right- A_∞ - B -module structure on M .*

Remark 3. *Every A_∞ -algebra (A, d_A) is an A_∞ - A - A -bimodule with A_∞ -bimodule structure given by the Taylor components $\bar{d}_A^{k,l} : A[1]^{\otimes k} \otimes A[1] \otimes A[1]^{\otimes l} \rightarrow A[1]$, with*

$$\bar{d}_A^{k,l} := \bar{d}_A^{k+l+1}.$$

Definition 9. *Let (M, d_M) and (N, d_N) be two A_∞ - A - B -bimodules, with $\mathcal{B}(M) = T(A[1]) \otimes M[1] \otimes T(B[1])$, and similarly for $\mathcal{B}(N)$. A morphism of A_∞ - A - B -bimodules is a morphism $F \in \text{Hom}_{\mathbf{bGK}}^{0,0}(\mathcal{B}(M), \mathcal{B}(N))$ of $\mathcal{B}(A)$ - $\mathcal{B}(B)$ -codifferential- counital bicomodules s.t.*

$$F \circ d_M = d_N \circ F.$$

Any A_∞ - A - B -bimodule morphism F is uniquely determined by its Taylor components $\bar{F}^{k,l} : A[1]^{\otimes k} \otimes M[1] \otimes B[1]^{\otimes l} \rightarrow N[1]$, $k, l \geq 0$. Explicitly

$$F^{k,l} = \sum_{s_3=0}^k \sum_{s_4=0}^l 1^{\otimes k-s_3} \otimes \bar{F}^{s_3,s_4} \otimes 1^{\otimes l-s_4},$$

where $F^{k,l} := F|_{A[1]^{\otimes k} \otimes M[1] \otimes B[1]^{\otimes l}}$. If A , resp. B are curved with curvature \bar{d}_A^0 , resp. \bar{d}_B^0 , then $F^{0,0}$ does not commute with $\bar{d}_M^{0,0}$ and $\bar{d}_N^{0,0}$ (which are not differentials).

4.0.4. Units in A_∞ -modules. Let (A, d_A) be a strictly unital A_∞ -algebra with unit 1_A and (M, d_M) a left A_∞ - A module. We introduce the desuspended maps

$$\bar{d}_M^l = -s \circ d_l^M \circ (s^{-1})^{\otimes l}, \quad l \geq 0.$$

Definition 10. *The module (M, d_M) is strictly unital if*

$$d_1^M(1_A, m) = m,$$

for every $m \in M$ and $d_n^M(a_1, \dots, a_n, m) = 0$ for $n \geq 2$ with $a_i = 1_A$ for some $i = 1, \dots, n$.

Similar considerations hold for right A_∞ -modules. A strictly unital morphism of strictly unital A_∞ -modules is an A_∞ -morphism F s.t.

$$F^n(a_1 | \dots | a_n | m) = 0, \quad n \geq 2$$

with $a_i = 1_A$ for some $i = 1, \dots, n$ and $F^1(1_A | m) = -sm$.

Similar definitions hold for unital A_∞ -bimodules over strictly unital A_∞ -algebras.

4.0.5. Homotopies of strictly unital A_∞ -modules. Let A be a strictly unital A_∞ -algebra and (M, d_M) , (N, d_N) be strictly unital A_∞ - A -modules. Let $f, g : M \rightarrow N$ be morphisms of A_∞ - A -modules; we say that M and N are A_∞ -homotopy equivalent (alternatively: A_∞ -homotopic) if there exists an A_∞ -homotopy between them, i.e. a bidegree $(-1, 0)$ morphism $H \in \text{Hom}_{\mathbf{bGK}}^{-1,0}(M[1] \otimes T(A[1]), N[1] \otimes T(A[1]))$ of counital $T(A[1])$ -comodules, s.t.

$$f_{\bar{h}} - g_{\bar{h}} = d_{N_{\bar{h}}} \circ H_{\bar{h}} + H_{\bar{h}} \circ d_{M_{\bar{h}}}.$$

4.0.6. *The tensor product of A_∞ -bimodules.* We consider now three A_∞ -algebras (A, d_A) , (B, d_B) and (C, d_C) . Furthermore, we introduce an A_∞ - A - B -bimodule (K_1, d_{K_1}) and an A_∞ - B - C -bimodule (K_2, d_{K_2}) .

Definition 11. *The tensor product $K_1 \underline{\otimes}_B K_2$ of K_1 and K_2 over B is the object*

$$K_1 \underline{\otimes}_B K_2 = K_1 \otimes T(B[1]) \otimes K_2$$

in $\mathbf{bG}_\mathbb{K}$.

Proposition 1 ([18]). *$K_1 \underline{\otimes}_B K_2$ is endowed with an A_∞ - A - C -bimodule structure given by the codifferential $d_{K_1 \underline{\otimes}_B K_2}$ with Taylor components $\bar{d}_{K_1 \underline{\otimes}_B K_2}^{m,n}$ given by*

$$\begin{aligned} & \bar{d}_{K_1 \underline{\otimes}_B K_2}^{m,n}(a_1 | \cdots | a_m | k_1 \otimes (b_1 | \cdots | b_q) \otimes k_2 | c_1 | \cdots | c_n) = 0, \quad m, n > 0 \\ & \bar{d}_{K_1 \underline{\otimes}_B K_2}^{m,0}(a_1 | \cdots | a_m | k_1 \otimes (b_1 | \cdots | b_q) \otimes k_2) = \\ & \sum_{l=0}^q s \left(s^{-1} (\bar{d}_{K_1}^{m,l}(a_1 | \cdots | a_m | k_1 | b_1 | \cdots | b_l)) \otimes (b_{l+1} | \cdots | b_q) \otimes k_2 \right), \quad m > 0 \\ & \bar{d}_{K_1 \underline{\otimes}_B K_2}^{0,n}(k_1 \otimes (b_1 | \cdots | b_q) \otimes k_2 | c_1 | \cdots | c_n) = \\ & (-1)^{|k_1| + \sum_{j=1}^q (|b_j| - 1)} \sum_{l=0}^q s \left(k_1 \otimes (b_1 | \cdots | b_l) \otimes s^{-1} (\bar{d}_{K_2}^{q-l,n}(b_{l+1} | \cdots | b_q | k_2 | c_1 | \cdots | c_n)) \right), \quad n > 0, \\ & \bar{d}_{K_1 \underline{\otimes}_B K_2}^{0,0}(s(k_1 \otimes (b_1 | \cdots | b_q) \otimes k_2)) = \\ & \sum_{l=0}^q s \left(s^{-1} (d_{K_2}^{0,l}(k_1 | b_1 | \cdots | b_l) \otimes (b_{l+1} | \cdots | b_q) \otimes k_2) \right) + \\ & \sum_{\substack{0 \leq l \leq q \\ 0 \leq p \leq q-l}} (-1)^{(|k_1| - 1) + \sum_{j=1}^l (|b_j| - 1)} s(k_1 \otimes (b_1 | \cdots | b_l | \bar{d}_B^p(b_{l+1} | \cdots | b_{l+p}) | \cdots | b_q) \otimes k_2) + \\ (6) \quad & (-1)^{|k_1| + \sum_{j=1}^q (|b_j| - 1)} \sum_{l=0}^q s \left(k_1 \otimes (b_1 | \cdots | b_l) \otimes s^{-1} (\bar{d}_{K_2}^{q-l,0}(b_{l+1} | \cdots | b_q | k_2)) \right). \end{aligned}$$

Corollary 1. *Let K_1 be an A_∞ - A - B -bimodule, K_2 an A_∞ - B - C -bimodule and K_3 an A_∞ - C - D -bimodule. The tensor product of A_∞ -bimodules is associative, i.e. there exists a strict A_∞ - A - D -bimodule morphism*

$$\Theta : (K_1 \underline{\otimes}_B K_2) \underline{\otimes}_C K_3 \rightarrow K_1 \underline{\otimes}_B (K_2 \underline{\otimes}_C K_3)$$

which induces an isomorphism of objects in $\mathbf{bG}_\mathbb{K}$.

4.0.7. *The A_∞ -bar constructions of an A_∞ -bimodule.* We consider two A_∞ -algebras (A, d_A) , (B, d_B) and an A_∞ - A - B -bimodule (M, d_M) . We recall that A can be canonically endowed with an A_∞ - A - A -bimodule structure; see Remark 11. Same holds for B , with due changes.

Definition 12. *The A_∞ - A - B -bimodule $(A \underline{\otimes}_A M, d_{A \underline{\otimes}_A M})$ is called the A_∞ -bar construction of (M, d_M) as left A_∞ - A -module. Similarly, the A_∞ - A - B -bimodule $(M \underline{\otimes}_B B, d_{M \underline{\otimes}_B B})$ is called the A_∞ -bar construction of (M, d_M) as right A_∞ - B -module.*

By definition, both $A \underline{\otimes}_A M$ and $M \underline{\otimes}_B B$ are A_∞ - A - B -bimodules. Let A and B be unital algebras and M an A - B -bimodule. Then $A \underline{\otimes}_A M$ is the bar resolution of M as left A -module. Similarly, $M \underline{\otimes}_B B$ is the bar resolution of M as right B -module.

Proposition 2 ([18]). *Let (A, d_A) , (B, d_B) be A_∞ -algebras and (M, d_M) be an A_∞ - A - B -bimodule. There exists a natural morphism*

$$\mu : A \underline{\otimes}_A M \rightarrow M,$$

of A_∞ - A - B -bimodules. If A , B are both flat, and A , M are left unital as A_∞ - A -module, then the morphism μ is a quasi-isomorphism.

4.0.8. *On the A_∞ -bar construction: a remark.* We continue our analysis of the A_∞ -bar constructions and the morphisms

$$\mu_A : A \underline{\otimes}_A K \rightarrow K, \quad \mu_B : K \underline{\otimes}_B B \rightarrow K$$

of strictly unital A_∞ - A - B -bimodules introduced in the above subsection. In the following lemma we restrict to the case of augmented associative algebras with zero differentials as they will appear later on.

Lemma 3. *Let (A, d_A) and (B, d_B) be augmented associative algebras with zero differential and (K, d_K) be a strictly unital A_∞ - A - B -bimodule.*

- *There exists strictly unital quasi-isomorphisms*

$$K \rightarrow A \underline{\otimes}_A K, \quad K \rightarrow K \underline{\otimes}_B B,$$

of A_∞ - A - B -bimodules.

Proof. We denote by

$$A_+ := \ker \epsilon_A, \quad B_+ := \ker \epsilon_B$$

the augmentation ideals in A , resp. B , denoting by ϵ_A resp. ϵ_B the augmentation maps on A , resp. B . We recall that the augmentation maps are morphisms of algebras. So the augmentation ideals are subalgebras.

We prove the first statement. The second is similar. Let

$$A \underline{\otimes}_{A_+} K = \bigoplus_{n \geq 0} A \otimes A_+[1]^{\otimes n} \otimes K,$$

be the normalized bar resolution of K . $A \underline{\otimes}_{A_+} K$ is a strictly unital A_∞ - A - B -bimodule. There exists a strict quasi-isomorphism

$$\mathcal{I} : A \underline{\otimes}_{A_+} K \rightarrow A \underline{\otimes}_A K$$

of A_∞ - A - B -bimodules; it is the natural inclusion. The quasi-isomorphism $K \rightarrow A \underline{\otimes}_A K$ is the composition

$$K \xrightarrow{\bar{\Phi}} A \underline{\otimes}_{A_+} K \xrightarrow{\mathcal{I}} A \underline{\otimes}_A K$$

where the (bidegree $(0,0)$) morphism $\bar{\Phi}$ is given as follows. Its (n,m) -th Taylor component $\bar{\Phi}_{n,m} : A[1]^{\otimes n} \otimes K[1] \otimes B[1]^{\otimes m} \rightarrow (A \underline{\otimes}_{A_+} K)[1]$ is simply

$$\bar{\Phi}_{n,m} = s \circ \Phi_{n,m} \circ (s^{-1})^{n+m+1}$$

with

$$\Phi_{n,m}(a_1, \dots, a_n, k, b_1, \dots, b_m) = 0 \text{ if } m \geq 1,$$

and

$$\Phi_{n,0}(a_1, \dots, a_n, k) = \begin{cases} (-1)^{\sum_{i=1}^n (|a_i|-1)} (1, a_1, \dots, a_n, k) & \text{if } a_i \in A_+, \text{ for all } i = 1, \dots, n. \\ 0 & \text{otherwise} \end{cases}$$

Note that $\Phi_{n,0}$ is of bidegree $(-n,0)$; Φ is strictly unital by construction. To check that

$$(7) \quad \bar{\Phi} \circ d_K = d_{A \underline{\otimes}_{A_+} K} \circ \bar{\Phi};$$

is straightforward. We need to consider (7) on all the possible strings of elements $(a_1 | \dots | a_m | k | b_1 | \dots | b_n) \in T(A[1]) \otimes K[1] \otimes T(B[1])$, $n, m \geq 0$ paying attention whether $(a_1 | \dots | a_n) \in A_+[1]^{\otimes n}$ or $sa_i \in \mathbb{K}[1]$, for some i . As $\Phi_{0,0}(1) = 1 \otimes 1$, then $\bar{\Phi}$ is a quasi-isomorphism. \square

Corollary 2. *Let A , B and K be as above.*

- *K and $A \underline{\otimes}_A K$ are homotopy equivalent as strictly unital A_∞ - A - B -bimodules.*
- *K and $K \underline{\otimes}_B B$ are homotopy equivalent as strictly unital A_∞ - A - B -bimodules.*

Proof. We prove the first statement; the second is analogous. We want to show that there exists a strictly unital A_∞ -homotopy $\bar{H} : K \rightarrow A \underline{\otimes}_A K$ of A_∞ - A - B -bimodules, s.t.

$$\begin{aligned} \bar{\Phi} \circ \mu_A &= 1 + d_{A \underline{\otimes}_A K} \circ \bar{H} + \bar{H} \circ d_{A \underline{\otimes}_A K}, \\ \mu_A \circ \bar{\Phi} &= 1, \end{aligned}$$

denoting by μ_A the A_∞ -morphism appearing in prop. 2 and by Φ the one appearing in lem. 3. The bidegree $(-1,0)$ Taylor components $\bar{H}_{m,n} : A[1]^{\otimes m} \otimes (A \underline{\otimes}_A K)[1] \otimes B[1]^{\otimes n} \rightarrow (A \underline{\otimes}_A K)[1]$ are given by $\bar{H}_{n,m} = 0$ if $m \geq 1$, and

$$\bar{H}_{n,0}(a_1 | \dots | a_n | (a, a'_1 | \dots | a'_q, k)) = \begin{cases} s(1, a_1 | \dots | a_n | a | a'_1 | \dots | a'_q, k) & \text{if } a_i \in A_+, \text{ for all } i = 1, \dots, n. \\ 0 & \text{otherwise} \end{cases}$$

$\mu_A \circ \bar{\Phi} = 1$ easily follows as K is strictly unital. The equality involving H is long to prove, but straightforward. By definition, the identity 1 is a strict and strictly unital A_∞ -morphism. \square

5. THE TRIPLE (A, K, B)

Let X be a finite dimensional vector space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In [3] it is shown that, choosing a pair (U, V) of subspaces in X , then it is possible to introduce a pair (A, B) of A_∞ -algebras associated to the subspaces themselves and an A_∞ -bimodule K associated to the intersection $U \cap V$. Choosing $(U, V) = (X, \{0\})$ we arrive at the pair of A_∞ -algebras

$$A = S(X^*), \quad B = \wedge(X).$$

A and B are objects in $\mathbf{bG}_{\mathbb{K}}$; let us discuss their bigrading. We put

$$A = \bigoplus_{i \geq 0} A_i, \quad A_i = A_i^0,$$

where A_i denotes the vector space of homogeneous polynomials of degree i . It follows that $A_0 = A_0^0 = \mathbb{K}$. A is concentrated in cohomological degree 0. The A_∞ -structure on A is encoded in a codifferential d_A whose only non trivial Taylor component is $\bar{d}_A^2 : A[1]^{\otimes 2} \rightarrow A[1]$. For the exterior algebra B we put

$$B = \bigoplus_{i \geq 0} B^i, \quad B^i = B_{-i}^i,$$

with $B_{-i}^i := \wedge^i X$. A bihomogeneous element $b \in B^i$ has bidegree $(i, -i)$. Also in this case $B_0 = B_0^0 = \mathbb{K}$. The A_∞ -structure on B is encoded in a codifferential d_B whose only non trivial Taylor component is $\bar{d}_B^2 : B[1]^{\otimes 2} \rightarrow B[1]$. In summary, the generators of B are bihomogeneous of bidegree $(1, -1)$; the dual generators in A are bihomogeneous with bidegree $(0, 1)$. Both A and B are augmented A_∞ -algebras with augmentation ideals $A_+ = \bigoplus_{i \geq 1} A_i^0$ and $B_+ = \bigoplus_{i \geq 1} B_{-i}^i$. Moreover

Proposition 3 ([3]). *Let X be a finite dimensional vector field over \mathbb{K} , $A = S(X^*)$, and $B = \wedge(X)$. There exists a one-dimensional strictly unital A_∞ - A - B -bimodule K which, as a left A -module and as a right B -module, is the augmentation module.*

The A_∞ - A - B -bimodule structure on K is specified by a codifferential d_K , with Taylor components $\bar{d}_K^{k,l} : A[1]^{\otimes k} \otimes K[1] \otimes B[1]^{\otimes l} \rightarrow K[1]$. We remind that, by definition, d_K (and so $\bar{d}_K^{k,l}$, for every $k, l \geq 0$) is of cohomological degree 1. The explicit construction in terms of Feynman diagrams implies that $\bar{d}_K^{k,l}(a_1 \dots |a_k|1|b_1| \dots |b_l|)$ is non vanishing iff

$$(8) \quad \sum_{i=1}^k \deg a_i = \sum_{i=1}^l |b_i| = k + l - 1,$$

where $\deg a_i$ denotes the internal degree of the homogeneous polynomial $a_i \in A$ and $|b_i|$ the cohomological grading of $b_i \in B^{|b_i|}$. But (8) implies that $\bar{d}_K^{k,l}$ is of degree 0 w.r.t the internal grading on A , B and K , for every $k, l \geq 0$: we recall that suspension and desuspension do not shift the internal degree.

The explicit construction of the codifferential d_K implies that K is a strictly unital A_∞ - A - B -bimodule.

5.0.9. *On the Keller condition for (A, K, B) .* We return to a more general setting.

Definition 13. *Let (A, d_A) and (B, d_B) be flat A_∞ -algebras and (K, d_K) be a right A_∞ - B -module. We set $\mathcal{R}(K) := K[1] \otimes T(B[1])$. $(\underline{\text{End}}_B(K), d_{\underline{\text{End}}_B(K)})$ is the flat A_∞ -algebra defined as follows. As bigraded object*

$$\underline{\text{End}}_B(K) := \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}(\mathcal{R}(K), K[1]);$$

the codifferential $d_{\underline{\text{End}}_B(K)}$ has non trivial Taylor components

$$(9) \quad \begin{aligned} \bar{d}_{\underline{\text{End}}_B(K)}^1 &= -s \circ \partial \circ s^{-1}, \quad \partial(\varphi) = (-1)^{|\varphi|+1}(\varphi \circ d_K) + d_K \circ \varphi, \\ \bar{d}_{\underline{\text{End}}_B(K)}^2(\varphi|\psi) &= (-1)^{|\varphi|}s(\varphi \circ \psi). \end{aligned}$$

We can define $(\underline{\text{End}}_A(K), d_{\underline{\text{End}}_A(K)})$ almost *verbatim*.

Proposition 4 ([15],[3]). *Let (A, d_A) and (B, d_B) be flat A_∞ -algebras and (K, d_K) be a right A_∞ - B -module. K is an A_∞ - A - B -bimodule² if and only if there exists a morphism*

$$L_A : A \rightarrow \underline{\text{End}}_B(K)$$

of A_∞ -algebras.

²We define a codifferential D_K s.t. $\bar{D}_K^{0,l} = \bar{d}_K^l$, for every $l \geq 0$.

Proof. A detailed proof can be found in [3]; here we sketch it. Let (K, d_K) be endowed with an A_∞ - A - B -bimodule structure D_K s.t. $\bar{D}_K^{0,l} = \bar{d}_K^l$. The maps

$$(10) \quad L_A(a_1 | \dots | a_k) \in \underline{\text{End}}_B(K)[1], \quad L_A(a_1 | \dots | a_k) := s \circ \mathcal{L}_A(a_1 | \dots | a_k)$$

with $\mathcal{L}_A(a_1 | \dots | a_k)$ of bidegree $(1, 0)$ given by

$$(11) \quad \mathcal{L}_A(a_1 | \dots | a_k)(1|b_1 | \dots | b_q) := \bar{D}_K^{k,l}(a_1 | \dots | a_k | 1|b_1 | \dots | b_q),$$

are the Taylor components of an A_∞ -algebra morphism $L_A : A \rightarrow \underline{\text{End}}_B(K)$, for every $(a_1 | \dots | a_k) \in A[1]^{\otimes k}$, $(1|b_1 | \dots | b_q) \in K[1] \otimes B[1]^{\otimes q}$ and $k \geq 1$, $q \geq 0$. Viceversa, let $L_A : A \rightarrow \underline{\text{End}}_B(K)$ be an A_∞ -algebra morphism with Taylor components as in (10). Then the maps $\bar{D}_K^{k,l}$ in (11) are the Taylor components of a codifferential D_K on $T(A[1]) \otimes K[1] \otimes T(B[1])$, extending the given right A_∞ - B -module structure on K . \square

We call L_A in prop. 4 the derived left A -action. A similar statement can be proved in the case of the derived right B -action, i.e. the A_∞ -algebra morphism $R_B : B^{op} \rightarrow \underline{\text{End}}_A(K)$ with obvious Taylor components. The A_∞ -algebra B^{op} has A_∞ -structure canonically induced by the one on B , but the signs are not trivial. We refer to [27] for all details.

Definition 14 ([10]). *Let (A, d_A) and (B, d_B) be flat A_∞ -algebras and (K, d_K) be an A_∞ - A - B -bimodule. The triple (A, K, B) satisfies the Keller condition if the derived actions*

$$L_A : A \rightarrow \underline{\text{End}}_B(K),$$

and

$$R_B : B^{op} \rightarrow \underline{\text{End}}_A(K),$$

are quasi-isomorphism of A_∞ -algebras.

5.0.10. *The Keller condition for the triple (A, K, B) .* Let (A, K, B) be the triple of bigraded A_∞ -structures given in section 5. The bigrading on the triple (A, K, B) is such that

$$\underline{\text{End}}_B^{i,j}(K) = \begin{cases} 0 & i + j < 0 \\ \text{Hom}_{\mathbf{bG}_K}^{0,0}(K[1] \otimes B[1]^{\otimes i+j}, K[1][i](j)) & i + j \geq 0 \end{cases}$$

Note that $\underline{\text{End}}_B^{0,0}(K) \cong \mathbb{K}$ and $\bar{d}_K^{0,1} \in \underline{\text{End}}_B^{2,0}(K)$. Similar considerations hold for $\underline{\text{End}}_A(K)$. The derived left action L_A preserves the internal grading, by definition. Moreover, for every $k \geq 1$ and $(a_1 | \dots | a_k) \in A[1]^{\otimes k}$, then $L_A(a_1 | \dots | a_k)$ is an element of $\underline{\text{End}}_B^{n,m}(K)$, with $(n, m) := (-k + 1, \sum_{i=1}^k \deg a_i)$.

For any $l \geq 0$ and $(1|b_1 | \dots | b_l) \in (K[1] \otimes B[1]^{\otimes l})_b^a$, with $(a, b) = (-1 + \sum_{i=1}^l |b_i| - l, -\sum_{i=1}^l |b_i|)$, we have

$$L_A(a_1 | \dots | a_k)(1|b_1 | \dots | b_l) := d_K^{k,l}(a_1 | \dots | a_k | 1|b_1 | \dots | b_l) \in (K[1])_{m+b}^{n+a}.$$

This implies that

$$n + a + 1 = 0 \Rightarrow \sum_{i=1}^l |b_i| = k + l - 1, \quad m + b = 0 \Rightarrow \sum_{i=1}^k \deg a_i = \sum_{i=1}^l |b_i|.$$

In other words, the wordlength l is uniquely determined by the constraint $l = 1 - k + \sum_{i=1}^k \deg a_i$, for any choice of $(b_1 | \dots | b_l) \in (B[1])^{\otimes l}$ as above. This analysis applies to R_B , with due changes. In [3] it is shown the important

Proposition 5. *The triple (A, K, B) given in section 5 is s.t. the derived left A -action L_A and the derived right B -action R_B are quasi-isomorphism of strictly unital A_∞ -algebras.*

As in the proof of proposition 4 we introduce the notation

$$\mathcal{L}_A(a_1 | \dots | a_n) \in \underline{\text{End}}_B^{r,m}(K), \quad \mathcal{L}_A(a_1 | \dots | a_n)(1|b_1 | \dots | b_q) = d_K^{n,q}(a_1 | \dots | a_n | 1|b_1 | \dots | b_q),$$

i.e. $L_A(a_1 | \dots | a_n) = s \circ \mathcal{L}_A(a_1 | \dots | a_n)$ and $r = \sum_{i=1}^n (|a_i| - 1) + 1$, $m = \sum_{i=1}^n \deg a_i$. We note that $\mathcal{L}_A(a_1 | \dots | a_n)$ is of cohomological degree $+1$. $A = S(X^*)$ is canonically endowed with a strictly unital A_∞ - A - A -bimodule structure \tilde{d}_A whose non trivial Taylor components are $\tilde{d}_A^{(1,0)} = \tilde{d}_A^{(0,1)} = \tilde{d}_A^2$.

Proposition 6. *There exists a strictly unital A_∞ - A - A -bimodule structure $d_{\underline{\text{End}}_B(K)}$ on $\underline{\text{End}}_B(K)$ such that the derived action L_A descends to a quasi-isomorphisms of strictly unital A_∞ - A - A -bimodules. $d_{\underline{\text{End}}_B(K)}$ has Taylor components*

$$\begin{aligned} \bar{d}_{\underline{\text{End}}_B(K)}^{0,0} &= -s \circ \partial_{\underline{\text{End}}_B(K)} \circ s^{-1}, \\ \bar{d}_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi) &= s \circ D_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi), \quad (n \geq 1) \\ \bar{d}_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | a_1 | \dots | a_m) &= s \circ D_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | a_1 | \dots | a_m), \quad (m \geq 1) \end{aligned}$$

with

$$\begin{aligned} \partial_{\underline{\text{End}}_B(K)}(\varphi) &= (-1)^{|\varphi|+1} \varphi \circ d_K + d_K \circ \varphi, \\ D_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi) &= (-1)^{\sum_{i=1}^n (|a_i|-1)-1} \mathcal{L}_A(a_1 | \dots | a_n) \circ \varphi, \\ D_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | a_1 | \dots | a_m) &= (-1)^{|\varphi|} \varphi \circ \mathcal{L}_A(a_1 | \dots | a_m), \end{aligned}$$

and $\bar{d}_{\underline{\text{End}}_B(K)}^{n,m} = 0$, otherwise.

Proof. See Appendix A. □

It can also be verified that the derived right- B action R_B descends to a quasi-isomorphism of A_∞ - B^{op} - B^{op} -bimodules.

6. A_∞ -MORITA THEORY

6.0.11. *On thm. 5.7. in [27].* In this section we study the A_∞ -Morita theory for the triple (A, K, B) . Our approach to the Morita equivalence is purely A_∞ ; all we need is the A_∞ - A - B -bimodule structure on K we described in the previous section to prove the equivalence of certain triangulated subcategories of A_∞ -modules in the derived categories $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$ of A and B . The functors giving the equivalences are defined through the A_∞ -tensor product of A_∞ modules and bimodules. The formalism is quite simple, using the associativity of the A_∞ -tensor product. The main advantage in using such “pure” A_∞ -approach is represented by the fact that the computations which follow are all explicit; the quasi-isomorphisms of A_∞ -bimodules which are the core of the equivalences are induced by the Keller condition on (A, K, B) .

6.0.12. *On some bigraded A_∞ -modules.* Let M be an A_∞ - A - B -bimodule and N be an A_∞ - B - C -bimodule, where A , B and C are A_∞ -algebras. We have already introduced the A_∞ - A - B -bimodule $(\mathcal{B}_B(M), d_{\mathcal{B}_B(M)})$, where $\mathcal{B}_B(M) := M \otimes_B B$, calling it the A_∞ -bar construction of M as right A_∞ - B -module. It is an A_∞ -right- B -module. If B is a differential bigraded algebra, then $\mathcal{B}_B(M)$ is a right- B -module. Note that A and B are not necessarily augmented.

Similarly, $({}_B\mathcal{B}(N), d_{{}_B\mathcal{B}(N)})$, with ${}_B\mathcal{B}(N) := B \otimes_B N$, is the A_∞ -bar construction of N as left A_∞ - B -module. It is an A_∞ -left- B -module. If B is a differential bigraded algebra, then $\mathcal{B}_B(M)$ is a left- B -module.

The following lemma is almost tautological, but it is helpful to fix notation.

Lemma 4. *Let A , C be flat A_∞ -algebras, B be a unital associative algebra and (N, d_N) be A_∞ - B - C -bimodule. If (M, d_M) is an A_∞ - A - B -bimodule such that $\bar{d}_M^{k,l} = 0$ if $(k, l) \neq (0, 0), (0, 1), (k, 0)$ and it is unital as right B -module, then there exists a strict A_∞ - A - C -bimodule isomorphism*

$$(12) \quad M \otimes_B N \equiv M \otimes_B {}_B\mathcal{B}(N).$$

The A_∞ - A - C -bimodule $M \otimes_B {}_B\mathcal{B}(N)$ in lem. 4 is given as follows. As bigraded object we have

$$(M \otimes_B {}_B\mathcal{B}(N))_j^i := \bigoplus_{\substack{i_1+i_2=i, \\ j_1+j_2=j}} M_{j_1}^{i_1} \otimes_B \mathcal{B}(N)_{j_2}^{i_2} / Q_j^i,$$

where $Q_j^i = \bigoplus_{i_1+i_2=i, j_1+j_2=j} Q \cap (M_{j_1}^{i_1} \otimes_B \mathcal{B}(N)_{j_2}^{i_2})$ and Q denotes the submodule in $M \otimes_B \mathcal{B}(N)$ generated by elements of the form $m \cdot b \otimes \tilde{B} - m \otimes b \cdot \tilde{B}$, with $m \in M$, $b \in B$ and $\tilde{B} \in {}_B\mathcal{B}(N)$. $M \otimes_B {}_B\mathcal{B}(N)$ is endowed with an A_∞ - A - C -bimodule structure given by a codifferential $d_{M \otimes_B {}_B\mathcal{B}(N)}$ with Taylor components

$$(13) \quad \bar{d}_{M \otimes_B {}_B\mathcal{B}(N)}^{0,0} = -s \circ D^{0,0} \circ s, \quad \bar{d}_{M \otimes_B {}_B\mathcal{B}(N)}^{n,0}, \quad \bar{d}_{M \otimes_B {}_B\mathcal{B}(N)}^{0,m},$$

with $n, m \geq 1$, s.t.

$$\begin{aligned} \mathcal{D}^{0,0}(m \otimes_B \tilde{B}) &= s^{-1}(\bar{d}_M^{0,0}(sm)) \otimes_B \tilde{B} + (-1)^{|m|} m \otimes_B s^{-1}(\bar{d}_{B\mathcal{B}(N)}^{0,0}(s\tilde{B})), \\ \bar{d}_{M \otimes_B B\mathcal{B}(N)}^{r,0}(a_1 | \dots | a_r | (m \otimes_B \tilde{B})) &= s(s^{-1}(\bar{d}_M^{r,0}(a_1 | \dots | a_r | m)) \otimes_B \tilde{B}), \\ \bar{d}_{M \otimes_B B\mathcal{B}(N)}^{0,m}((m \otimes_B (b \otimes b_1 | \dots | b_q \otimes n)) | c_1 | \dots | c_m) &= (-1)^{|m|+|b|+\sum_{i=1}^q(|b_i|-1)} \\ &\sum_{q'=0}^q s(m \otimes_B (b \otimes b_1 | \dots | b_{q'} \otimes s^{-1}(\bar{d}_N^{q',n}(b_{q'+1} | \dots | b_q | n | c_1 | \dots | c_m))), \end{aligned}$$

and zero otherwise.

Remark 4. Exchanging the role of M and N in lemma 4 we can describe the strict A_∞ - A - C -bimodule isomorphism

$$M \underline{\otimes}_B N \equiv \mathcal{B}_B(M) \otimes_B N.$$

Remark 5. In what follows we only consider the triple (A, K, B) of bigraded A_∞ -objects with A_∞ -algebras (A, d_A) and (B, d_B) s.t. $A = S(X^*)$, $B = \wedge(X)$ and A_∞ -bimodule (K, d_K) , $K = \mathbb{K}$.

6.0.13. On the right derived A_∞ -module \underline{K} . Let $\mathcal{B}_B(K) := K \underline{\otimes}_B B$ denote the bar construction of K as right B -module. By definition,

$$\mathcal{B}_B(K)_j^i := \bigoplus_{q \geq 0} (K \otimes B[1]^{\otimes q} \otimes B)_j^i.$$

and

$$\mathcal{B}_B(K)_j^i = \begin{cases} 0 & i + j > 0, \\ K \otimes (B[1]^{\otimes -(i+j)} \otimes B)_j^i & i + j \leq 0. \end{cases}$$

We have also the isomorphism $\mathcal{B}_B(K) \cong M \otimes B$ in $\mathbf{bG}_{\mathbb{K}}$, where $M_j^i = \bigoplus_{q \geq 0} K \otimes (B[1]^{\otimes q})_j^i = K \otimes (B[1]^{\otimes -(i+j)})_j^i$.

Definition 15. The right derived dual module \underline{K} of K is the object

$$(14) \quad \underline{K} = \mathbf{Hom}_B(\mathcal{B}_B(K), B).$$

in $\mathbf{bG}_{\mathbb{K}}$.

We recall that, for every pair M, N of right B -modules, then $\mathbf{Hom}_B(M, N)$ is the object in $\mathbf{bG}_{\mathbb{K}}$ with bihomogeneous components $\mathbf{Hom}_B^{i,j}(M, N) = \{\varphi \in \mathbf{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{i,j}(M, N), \varphi \text{ right } B\text{-linear}\}$.

Lemma 5. \underline{K} can be endowed with a strictly unital A_∞ - B - A -bimodule structure $d_{\underline{K}}$ with Taylor components given by

$$\begin{aligned} \bar{d}_{\underline{K}}^{0,0} &= -s \circ \partial_{\underline{K}} \circ s^{-1}, \\ \bar{d}_{\underline{K}}^{1,0}(b|\varphi) &= s \circ D_{\underline{K}}^{1,0}(b|\varphi), \\ \bar{d}_{\underline{K}}^{0,m}(\varphi|a_1 | \dots | a_m) &= s \circ D_{\underline{K}}^{0,m}(\varphi|a_1 | \dots | a_m), \end{aligned}$$

with

$$\begin{aligned} \partial_{\underline{K}}(\varphi) &= (-1)^{|\varphi|} \varphi \circ \bar{d}_{\mathcal{B}_B(K)}^{0,0}, \\ D_{\underline{K}}^{1,0}(b|\varphi)(1, b_1 | \dots | b_q, b') &= (-1)^{|b|} b \cdot \varphi(1, b_1 | \dots | b_q, b'), \\ D_{\underline{K}}^{0,m}(\varphi|a_1 | \dots | a_m)(1, b_1 | \dots | b_q, b') &= \\ &(-1)^{|\varphi|-1+\sum_{i=1}^m(|a_i|-1)} \sum_{q'=0}^q \varphi(s^{-1} \bar{d}_K^{m,q'}(a_1 | \dots | a_m | 1 | b_1 | \dots | b_{q'}), b_{q'+1} | \dots | b_q, b), \end{aligned}$$

and $\bar{d}_{\mathcal{B}_B(K)}^{0,0} = -s \circ \bar{d}_{\mathcal{B}_B(K)}^{0,0} \circ s^{-1}$, $\bar{d}_{\underline{K}}^{n,m} = 0$, otherwise.

Corollary 3. $(\underline{K}, d_{\underline{K}})$ is a strictly unital differential bigraded left B -module; we have a strict isomorphism

$$K \underline{\otimes}_B \underline{K} \equiv \mathcal{B}_B(K) \otimes_B \underline{K}$$

of strictly unital A_∞ - A - A -bimodules.

6.0.14. On the quasi-isomorphism $A \rightarrow K \underline{\otimes}_B \underline{K}$.

Definition 16. $\mathbf{End}_B(\mathcal{B}_B(K))$ is the object in $\mathbf{bG}_\mathbb{K}$ with bihomogeneous components

$$(15) \quad \mathbf{End}_B^{i,j}(\mathcal{B}_B(K)) = \mathbf{Hom}_B^{0,0}(K \underline{\otimes}_B B, (K \underline{\otimes}_B B)[i]\langle j \rangle).$$

Lemma 6. $\mathbf{End}_B(\mathcal{B}_B(K))$ can be endowed with a strictly unital A_∞ - A - A -bimodule structure $\mathbf{d}_{\mathbf{End}_B(\mathcal{B}_B(K))}$ with Taylor components

$$\begin{aligned} \bar{\mathbf{d}}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{0,0} &= -s \circ \partial_{\mathbf{End}_B(\mathcal{B}_B(K))} \circ s^{-1}, \\ \bar{\mathbf{d}}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{l,0}(a_1 | \dots | a_l | \varphi) &= s \circ \mathbf{D}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{l,0}(a_1 | \dots | a_l | \varphi), \\ \bar{\mathbf{d}}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{0,m}(\varphi | a_1 | \dots | a_m) &= s \circ \mathbf{D}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{0,m}(\varphi | a_1 | \dots | a_m), \end{aligned}$$

with

$$\begin{aligned} \partial_{\mathbf{End}_B(\mathcal{B}_B(K))}(\varphi) &= (-1)^{|\varphi|} \varphi \circ \bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{0,0} - \bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{0,0} \circ \varphi, \\ \mathbf{D}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{l,0}(a_1 | \dots | a_l | \varphi)(1, b_1 | \dots | b_q, b) &= \\ (-1)^{\sum_{i=1}^l (|a_i| - 1) - 1} s^{-1}(\bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{l,0}(a_1 | \dots | a_l | \varphi(1, b_1 | \dots | b_q, b))), (l \geq 1) \\ \mathbf{D}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{0,m}(\varphi | a_1 | \dots | a_m)(1, b_1 | \dots | b_q, b) &= \\ (-1)^{|\varphi|} \varphi(\sum_{q'=0}^q s^{-1}(\mathbf{d}_K^{m,q'}(a_1 | \dots | a_m | 1 | b_1 | \dots | b_{q'}), b_{q'+1} | \dots | b_q, b), \end{aligned}$$

and $\bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{0,0} = -s \circ \bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{0,0} \circ s^{-1}$, where $\bar{\mathbf{d}}_{\mathcal{B}_B(K)}^{0,0}$ is given in proposition 1, $\bar{\mathbf{d}}_{\mathbf{End}_B(\mathcal{B}_B(K))}^{n,m} = 0$, otherwise.

Let $(\underline{\mathbf{End}}_B(K), \mathbf{d}_{\underline{\mathbf{End}}_B(K)})$ be the strictly unital A_∞ - A - A -bimodule described in prop 6.

We recall that the bar resolution $\mathcal{B}_B(K) = K \underline{\otimes}_B B$ is homotopy equivalent to K in $\mathbf{bG}_\mathbb{K}$ (but not as right bigraded B -modules); the maps giving such homotopy equivalence are the projection $p : K \underline{\otimes}_B B \rightarrow K$ and the inclusion $i : K \rightarrow K \underline{\otimes}_B B$, with $p(1, b) = b(0)$ and $p(1, b_1 | \dots | b_q, b) = 0$ for $q \geq 1$.

Proposition 7. $(\mathbf{End}_B(\mathcal{B}_B(K)), \mathbf{d}_{\mathbf{End}_B(\mathcal{B}_B(K))})$ and $(\underline{\mathbf{End}}_B(K), \mathbf{d}_{\underline{\mathbf{End}}_B(K)})$ are homotopy equivalent as strictly unital A_∞ - A - A -bimodules.

Proof. We define the strict (and strictly unital) morphism $\mathcal{H} : \mathbf{End}_B(\mathcal{B}_B(K)) \rightarrow \underline{\mathbf{End}}_B(K)$ of strictly unital A_∞ - A - A -bimodules, via $\mathcal{H} = s \circ H \circ s^{-1}$, where, for any $(i, j) \in \mathbb{Z}^2$, $H : \mathbf{End}_B^{i,j}(\mathcal{B}_B(K)) \rightarrow \underline{\mathbf{End}}_B^{i,j}(K)$ is the composition $H := (s \circ 1 \circ s^{-1}) \circ \mathcal{P} \circ \mathcal{I}$, denoting by \mathcal{I} and \mathcal{P} the morphisms

$$\begin{aligned} \mathbf{Hom}_B^{0,0}(\mathcal{B}_B(K), \mathcal{B}_B(K)[i]\langle j \rangle) &\xrightarrow{\mathcal{I}} \mathbf{Hom}_{\mathbf{bG}_\mathbb{K}}^{0,0}(K \otimes \mathbf{T}(B[1]), \mathcal{B}_B(K)[i]\langle j \rangle) \\ &\xrightarrow{\mathcal{P}} \mathbf{Hom}_{\mathbf{bG}_\mathbb{K}}^{0,0}(K \otimes \mathbf{T}(B[1]), K[i]\langle j \rangle) \xrightarrow{s \circ 1 \circ s^{-1}} \mathbf{Hom}_{\mathbf{bG}_\mathbb{K}}^{0,0}(K[1] \otimes \mathbf{T}(B[1]), (K[1])[i]\langle j \rangle), \end{aligned}$$

with $\mathcal{I}(\varphi)(1, b_1 | \dots | b_q) := \varphi(1, b_1 | \dots | b_q, 1)$, $\mathcal{P}(\psi) := p \circ \psi$. More explicitly, if $\varphi \in \mathbf{End}_B^{i,j}(\mathcal{B}_B(K))$, then

$$(16) \quad (H\varphi)(1 | b_1 | \dots | b_q) = s((\varphi^0(1, b_1 | \dots | b_q, 1))(0)),$$

denoting by $\varphi^0(1, b_1 | \dots | b_q, 1)$ the projection of $\varphi(1, b_1 | \dots | b_q, 1)$ onto $K \otimes B$.

To prove

$$\mathcal{H} \circ \mathbf{d}_{\mathbf{End}_B} = \mathbf{d}_{\underline{\mathbf{End}}_B(K)} \circ \mathcal{H}$$

is straightforward; the only issue is represented by the signs; all details are contained in [?]. \square

Proposition 8. $(K \underline{\otimes}_B \underline{K}, \mathbf{d}_{K \underline{\otimes}_B \underline{K}})$ and $(\mathbf{End}_B(\mathcal{B}_B(K)), \mathbf{d}_{\mathbf{End}_B(\mathcal{B}_B(K))})$ are strictly isomorphic as strictly unital A_∞ - A - A -bimodules.

Proof. We recall that $\mathcal{B}_B(K) = M \otimes B$ in $\mathbf{bG}_\mathbb{K}$. The strict isomorphism of A_∞ - A - A -bimodules

$$\mathcal{G} : \mathcal{B}_B(K) \otimes_B \underline{K} \rightarrow \mathbf{End}_B(\mathcal{B}_B(K)),$$

with $\mathcal{G} = s \circ G \circ s^{-1}$ is given as follows. The morphism G is defined by the commutative diagram

$$\begin{array}{ccc}
\mathcal{B}_B(K) \otimes_B \mathbf{Hom}_B(\mathcal{B}_B(K), B) & \xrightarrow{G} & \mathbf{End}_B(\mathcal{B}_B(K)) \\
\downarrow \tau_1 & & \parallel \\
(M \otimes B) \otimes_B \mathbf{Hom}_{\mathbf{bG}_{\mathbb{K}}}(M, B) & & \mathbf{End}_B(\mathcal{B}_B(K)) \\
\downarrow \tau_2 & & \downarrow \mathcal{I} \\
M \otimes \mathbf{Hom}_{\mathbf{bG}_{\mathbb{K}}}(M, B) & \xrightarrow{\underline{G}} & \mathbf{Hom}_{\mathbf{bG}_{\mathbb{K}}}(M, M \otimes B)
\end{array}$$

in $\mathbf{bG}_{\mathbb{K}}$, where

$$(17) \quad \underline{G}(m, \varphi)(m') := m \otimes \varphi(m'),$$

and $\tau_1, \tau_2, \mathcal{I}$ denote the obvious isomorphisms. Note the sign in $\tau_2((m \otimes b) \otimes_B \varphi) = (-1)^{|m|+|b|+|\varphi|} m \otimes b\varphi$.

More explicitly

$$(18) \quad G((m \otimes b) \otimes_B \varphi)(m' \otimes b') := (-1)^{|m|+|b|+|\varphi|} m \otimes b\varphi(m' \otimes b').$$

By definition, $G(Q_j^i) = 0$ for every $(i, j) \in \mathbb{Z}^2$, where Q_j^i is the submodule in $(\mathcal{B}_B(K) \otimes \mathbf{Hom}_B(\mathcal{B}_B(K), B))_j^i$ introduced in the proof of lemma 4. So G is well defined, as morphism in $\mathbf{bG}_{\mathbb{K}}$. Note that $\tau_2(Q_j^i) = 0$, as well. \underline{G} is an isomorphism in $\mathbf{bG}_{\mathbb{K}}$; so G is an isomorphism in $\mathbf{bG}_{\mathbb{K}}$ as well; in fact M is an object in $\mathbf{bG}_{\mathbb{K}}$ with finite dimensional bihomogeneous components $M_j^i = K \otimes (B[1]^{\otimes -(i+j)})_j^i$, for every $i, j \in \mathbb{Z}$. We finish the proof of proposition 8 by checking that G is a chain map and commutes with the left and right A_∞ - A -actions on $\mathcal{B}_B(K) \otimes_B \underline{K}$ and $\mathbf{End}_B(\mathcal{B}_B(K))$. The only issue is represented by the signs appearing in G and in the Taylor components of the codifferentials on $\mathcal{B}_B(K) \otimes_B \underline{K}$ and $\mathbf{End}_B(\mathcal{B}_B(K))$. In particular, the non trivial sign in (18) is necessary to prove compatibility between G and the right A_∞ -module structures, i.e.

$$\bar{d}_{\mathbf{End}_B(K)}^{0,m}(\mathcal{G}((m \otimes b) \otimes_B \varphi), a_1 | \dots | a_m) = \mathcal{G}(\bar{d}_{\mathcal{B}_B(K) \otimes_B \underline{K}}^{0,m}(((m \otimes b) \otimes_B \varphi) | a_1 | \dots | a_m),$$

with l.h.s. equal to (applying it on $m' \otimes b'$ with $m' \otimes b' = 1, b_1 | \dots | b_q \otimes b'$)

$$\sum_{q'=0}^q m \otimes b\varphi(s^{-1}(\bar{d}_{\underline{K}}^{m,q'}(a_1 | \dots | a_m | 1 | b_1 | \dots | b_q)), b_{q'} | \dots | b_q, b'),$$

and r.h.s. equal to

$$(-1)^{|m|+|b|} \mathcal{G}((m \otimes b) \otimes_B s^{-1}(\bar{d}_{\underline{K}}^{0,m}(\varphi | a_1 | \dots | a_m))) = (-1)^{|\varphi|-1+\sum_{i=1}^m(|a_i|-1)} m \otimes b \cdot s^{-1}(\bar{d}_{\underline{K}}^{0,m}(\varphi | a_1 | \dots | a_m)(1, b_1 | \dots | b_q, b')).$$

The Taylor components $\bar{d}_{\underline{K}}^{0,m}$ generate the sign $(-1)^{|\varphi|+\sum_{i=1}^m(|a_i|-1)-1}$; so we are done. \square

We summarize the results so far into

Corollary 4. $(K \otimes_B \underline{K}, d_{K \otimes_B \underline{K}})$ and $(\mathbf{End}_B(K), d_{\mathbf{End}_B(K)})$ are homotopy equivalent as strictly unital A_∞ - A - A -bimodules.

Proposition 9. *There exists a strictly unital quasi-isomorphism*

$$A \rightarrow K \otimes_B \underline{K}$$

of strictly unital A_∞ - A - A -bimodules.

Proof. Just compose the homotopy equivalence in the above corollary with the left derived action L_A . \square

6.0.15. *On the quasi-isomorphism $B \rightarrow \underline{K} \otimes_A K$.* Following the example of \underline{K} , we can introduce the left derived bimodule

$$\overline{K} = \mathbf{Hom}_A(A \otimes_A K, A).$$

As $A[1]$ is concentrated in cohomological degree -1 , then

$$A \otimes_A K = A \otimes N$$

in $\mathbf{bG}_{\mathbb{K}}$, with $N_j^i = 0$ if $i > 0$, $N_0^0 = K$ and

$$N_j^i = \bigoplus_{j_1 + \dots + j_{-i} = j} \overbrace{(A[1])_{j_1}^{-1} \otimes \dots \otimes (A[1])_{j_{-i}}^{-1}}^{-i\text{-times}}$$

for any $i < 0$. Every bihomogeneous component of N is finite dimensional. In what follows ${}_A \mathcal{B}(K) := A \otimes_A K$.

By definition, \overline{K} is a strictly unital A_∞ - B - A -bimodule with codifferential $d_{\overline{K}}$ whose Taylor components are given by

$$\bar{d}_{\overline{K}}^{0,0} = -s \circ \partial_{\overline{K}} \circ s^{-1}, \quad \bar{d}_{\overline{K}}^{k,0}(b_1 | \dots | b_k | \varphi) = s \circ D_{\overline{K}}^{k,0}(b_1 | \dots | b_k | \varphi), \quad \bar{d}_{\overline{K}}^{0,1}(\varphi | a) = s \circ D_{\overline{K}}^{0,1}(\varphi | a),$$

with

$$\begin{aligned} \partial_{\overline{K}}(\varphi) &= (-1)^{|\varphi|} \varphi \circ \bar{d}_{A\mathcal{B}(K)}^{0,0}, \\ D_{\overline{K}}^{k,0}(b_1 | \dots | b_k | \varphi)(a, a_1 | \dots | a_q, 1) &= (-1)^{(|\varphi|+|a|+\sum_{i=1}^q(|a_i|-1)+1)(\sum_{i=1}^k(|b_i|-1)+1)} \\ &\quad \sum_{q'=0}^q \varphi(a, a_1 | \dots | a_{q-q'} | s^{-1} \bar{d}_K^{q',k}(a_{q-q'+1} | \dots | a_q | 1 | b_1 | \dots | b_k)), \\ D_{\overline{K}}^{0,1}(\varphi | a')(m) &= (-1)^{|\varphi|+|a||m|} \varphi(m) \cdot a, \end{aligned}$$

where $\bar{d}_{A\mathcal{B}(K)}^{0,0} = -s \circ \bar{d}_{A\mathcal{B}(K)}^{0,0} \circ s^{-1}$ and $\bar{d}_{\overline{K}}^{n,m} = 0$, otherwise.

To check that $(\overline{K}, d_{\overline{K}})$ is a strictly unital A_∞ - B - A -bimodule is long but straightforward.

Definition 17. $\mathbf{End}_A(A\mathcal{B}(K))^{op}$ is the object in $\mathbf{bG}_{\mathbb{K}}$ with bihomogeneous components

$$\mathbf{End}_A^{i,j}(A\mathcal{B}(K))^{op} = \mathbf{Hom}_A^{0,0}(A \otimes_A K, (A \otimes_A K)[i]\langle j \rangle).$$

Lemma 7. $\mathbf{End}_A(A\mathcal{B}(K))^{op}$ can be endowed with a strictly unital A_∞ - B - B -bimodule structure $d_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}$ with Taylor components

$$\begin{aligned} \bar{d}_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{0,0} &= -s \circ \partial_{\mathbf{End}_A(A\mathcal{B}(K))^{op}} \circ s^{-1}, \\ \bar{d}_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{l,0}(b_1 | \dots | b_l | \varphi) &= s \circ D_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{l,0}(b_1 | \dots | b_l | \varphi), \\ \bar{d}_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{0,m}(\varphi | b_1 | \dots | b_m) &= s \circ D_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{0,m}(\varphi | b_1 | \dots | b_m), \end{aligned}$$

with

$$\begin{aligned} \partial_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}(\varphi) &= (-1)^{|\varphi|} \varphi \circ \bar{d}_{A\mathcal{B}(K)}^{0,0} - \bar{d}_{A\mathcal{B}(K)}^{0,0} \circ \varphi, \\ D_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{l,0}(b_1 | \dots | b_l | \varphi)(a, a_1 | \dots | a_q, 1) &= (-1)^{(|\varphi|+|a|+\sum_{i=1}^q(|a_i|-1)+1)(\sum_{i=1}^l(|b_i|-1)+1)} \\ &\quad \sum_{q'=0}^q \varphi(a, a_1 | \dots | a_{q'} | s^{-1} \bar{d}_K^{q-q',l}(a_{q'+1} | \dots | a_q | 1 | b_1 | \dots | b_l)), \quad (l \geq 1) \\ D_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{0,m}(\varphi | b_1 | \dots | b_m)(a, a_1 | \dots | a_q, 1) &= \\ (-1)^{(|a|+\sum_{i=1}^q(|a_i|-1))\sum_{i=1}^m(|b_i|-1)} \bar{d}_{A\mathcal{B}(K)}^{0,m}(\varphi(a, a_1 | \dots | a_q, 1) | b_1 | \dots | b_m), \quad (m \geq 1) \end{aligned}$$

and $\bar{d}_{A\mathcal{B}(K)}^{0,0} = -s \circ \bar{d}_{A\mathcal{B}(K)}^{0,0} \circ s^{-1}$, where $\bar{d}_{A\mathcal{B}(K)}^{0,0}$ is given in proposition 1, and $\bar{d}_{\mathbf{End}_A(A\mathcal{B}(K))^{op}}^{n,m} = 0$, otherwise.

Proposition 10. $(\overline{K} \otimes_A K, d_{\overline{K} \otimes_A K})$ and $(\mathbf{End}_A(A\mathcal{B}(K))^{op}, d_{\mathbf{End}_A(A\mathcal{B}(K))^{op}})$ are strictly isomorphic as strictly unital A_∞ - B - B -bimodules.

Proof. The proof is similar to the one of prop. 8, with due changes. □

Proposition 11. There exists a strictly unital quasi-isomorphism

$$B \rightarrow \overline{K} \otimes_A K$$

of strictly unital A_∞ - B - B -bimodules.

Proof. Just compose the homotopy equivalence in the above prop. with the right derived action R_B . □

6.0.16. A_∞ -Morita theory for the triple (A, K, B) .

6.0.17. *On the functors.* Let us consider the functors

$$F' : \mathbf{Mod}_\infty(A) \rightarrow \mathbf{Mod}_\infty^{strict}(B), \quad G' : \mathbf{Mod}_\infty(B) \rightarrow \mathbf{Mod}_\infty^{strict}(A),$$

given by

$$F'(M) := M \underline{\otimes}_A K, \quad G'(N) := N \underline{\otimes}_B \underline{K},$$

on objects $M \in \mathbf{Mod}_\infty(A)$ and $N \in \mathbf{Mod}_\infty(B)$, while on morphisms $f : M_1 \rightarrow M_2$ in $\mathbf{Mod}_\infty A$ and $g : N_1 \rightarrow N_2$ in $\mathbf{Mod}_\infty B$ we set

$$F'(f) := M_1 \underline{\otimes}_A K \rightarrow M_2 \underline{\otimes}_A K, \quad G'(g) := N_1 \underline{\otimes}_B \underline{K} \rightarrow N_2 \underline{\otimes}_B \underline{K},$$

with

$$F'(f) := (s^{-1} \circ F \circ s) \otimes 1, \quad G'(g) := (s^{-1} \circ G \circ s) \otimes 1.$$

We have denoted by $F : \mathcal{R}(M_1) \rightarrow \mathcal{R}(M_2)$, respectively $G : \mathcal{R}(N_1) \rightarrow \mathcal{R}(N_2)$, the unique lifting of f (resp. g) to a $T(A[1])$ -counital-comodule morphism, respectively a $T(B[1])$ -counital-comodule morphism. In this notation, $\mathcal{R}(M_1) := M_1[1] \otimes T(A[1])$ and similarly for $\mathcal{R}(N_1)$, with due changes. Let \mathcal{F} and \mathcal{G} be the functors given by the compositions

$$\mathcal{F} : \mathbf{Mod}_\infty(A) \xrightarrow{F'} \mathbf{Mod}_\infty^{strict}(B) \xrightarrow{i} \mathbf{Mod}_\infty(B),$$

and

$$\mathcal{G} : \mathbf{Mod}_\infty(B) \xrightarrow{G'} \mathbf{Mod}_\infty^{strict}(A) \xrightarrow{i} \mathbf{Mod}_\infty(A),$$

denoting by i the inclusion of the subcategories $\mathbf{Mod}_\infty^{strict}(A)$ (resp. $\mathbf{Mod}_\infty^{strict}(B)$) in $\mathbf{Mod}_\infty A$ (resp. $\mathbf{Mod}_\infty(B)$). We remark that $\mathbf{Mod}_\infty^{strict}(A)$ and $\mathbf{Mod}_\infty^{strict}(B)$ are not full subcategories.

If two morphisms f and g in $\mathbf{Mod}_\infty(A)$ are (A_∞) -homotopic, then we write $f \sim g$. An analogous notation holds true in $\mathbf{Mod}_\infty(B)$. If the homotopy between f and g is strict, then we write $f \sim_{strict} g$.

Lemma 8. a) *Let $f \sim g$ in $\mathbf{Mod}_\infty A$, resp. in $\mathbf{Mod}_\infty B$. Then*

$$F'(f) \sim_{strict} F'(g),$$

in $\mathbf{Mod}_\infty^{strict} B$, resp.

$$G'(f) \sim_{strict} G'(g),$$

in $\mathbf{Mod}_\infty^{strict} A$.

b) *The functors F' and G' send strictly unital homotopy equivalences to strict and strictly unital homotopy equivalences.*

Proof. Part a). Let $f, g : M \rightarrow N$ with $f \sim g$ in $\mathbf{Mod}_\infty(A)$, i.e. $f - g = d_N h + h d_M$, where $h : M \rightarrow N$ is a strictly unital A_∞ -homotopy. By definition, h is a degree -1 map with components $h_n : M[1] \otimes A[1]^{\otimes n} \rightarrow N[1]$, $n \geq 0$. We claim that $H : M \underline{\otimes}_A K \rightarrow N \underline{\otimes}_A K$, where

$$H_0 := (s^{-1} \circ h \circ s) \otimes 1$$

and $H_n = 0$ for $n > 0$, is a strict A_∞ -homotopy between $F'(f)$ and $F'(g)$, i.e.

$$(19) \quad F'(f) - F'(g) = d_{N \underline{\otimes}_A K} \circ H + H \circ d_{M \underline{\otimes}_A K}.$$

Eq. (19) is equivalent to

$$(20) \quad (F'(f) - F'(g))(s(m, a_1 | \dots | a_q, 1)) = (d_{N \underline{\otimes}_A K} \circ H + H \circ d_{M \underline{\otimes}_A K})(s(m, a_1 | \dots | a_q, 1)),$$

and

$$(21) \quad 0 = (d_{N \underline{\otimes}_A K} \circ H + H \circ d_{M \underline{\otimes}_A K})((m, a_1 | \dots | a_q, 1) | b_1 | \dots | b_l),$$

for every $q, l \geq 0$. Let us consider at first eq. (20); on the l.h.s. we have terms involving the Taylor components of the codifferential d_M on M and the A_∞ -homotopy h by the homotopy hypothesis $f \sim g$; all we need to prove is that on the r.h.s the terms involving the Taylor components of the codifferential d_K on K cancel. This is true because these terms appear in

$$\begin{aligned} & \sum_{q_1=0}^q \sum_{q_2=0}^{q_1+1} (-1)^{1+|m|+\sum_{i=1}^{q_2} (|a_i|-1)} ((h_{q_1}(m, a_1 | \dots | a_{q_1}) | a_{q_1+1} | \dots | a_{q-q_2}, d_K^{q_2}(a_{q-q_2+1} | \dots | a_q | 1)) + \\ & \sum_{q_1=0}^q \sum_{q_2=0}^{q_1+1} (-1)^{|m|+(|a_i|-1)} (h_{q_1}(m, a_1 | \dots | a_{q_1}), a_{q_1+1} | \dots | a_{q-q_2}, d_K^{q_2}(a_{q-q_2+1} | \dots | a_q | 1)) = 0, \end{aligned}$$

as the A_∞ -homotopy h has degree -1 . Eq. (21) is equivalent to

$$0 = \sum_{q_1=0}^q d_{N \otimes_A K}(s^{-1}(h_{q_1}(m|a_1| \dots |a_{q_1}))|a_{q_1+1}| \dots |a_q|1) + \\ \sum_{q_2=0}^q (-1)^{1+|m|+\sum_{i=1}^{q-q_2}(|a_i|-1)} H(m|a_1| \dots |a_{q-q_2}, d_K^{q_2,l}(a_{q-q_2+1}| \dots |a_q|1|b_1| \dots |b_l)),$$

which is verified by the same argument we used for eq. (20) and (21). The case $f \sim g$ in $\mathbf{Mod}_\infty(B)$ is similar.

Part **b)**. The morphism $f : M \rightarrow N$ is a homotopy equivalence in $\mathbf{Mod}_\infty A$ if there exists a morphism $g : N \rightarrow M$ in $\mathbf{Mod}_\infty(A)$ s.t. $f \circ g \sim 1$ and $g \circ f \sim 1$. We denote by $h_1 : N \rightarrow N$, resp. $h_2 : M \rightarrow M$ the A_∞ -homotopies between $f \circ g$ and 1_N , resp. $g \circ f$ and 1_M . We want to prove that

$$F'(f) \circ F'(g) = 1 + d_{N \otimes_A K} \circ H_1 + H_1 \circ d_{N \otimes_A K}, \\ F'(g) \circ F'(f) = 1 + d_{M \otimes_A K} \circ H_2 + H_2 \circ d_{M \otimes_A K},$$

with strict A_∞ -homotopies

$$H_i := (s^{-1} \circ h_i \circ s) \otimes 1,$$

for $i = 1, 2$. Using the proof of **a)** we get the statement. The case for G' is similar. \square

6.0.18. *On the derived categories.* In this section we introduce the derived categories $\mathbf{D}^\infty(A)$, respectively $\mathbf{D}^\infty(B)$, of right unital A_∞ -modules over A , respectively B , with strictly unital A_∞ -morphisms. Using the theory of closed model categories it is possible to prove

Theorem 6 (K. Lefevre-Hasegawa, [12]). *Let A be an augmented A_∞ -algebra³; quasi-isomorphisms in $\mathbf{Mod}_\infty(A)$ are homotopy equivalences of strictly unital A_∞ - A -modules.*

This results implies that

$$\mathbf{D}^\infty(A) = \mathbf{Mod}_\infty(A) / \sim,$$

and similarly for $\mathbf{D}^\infty(B)$. In this setting quasi-isomorphisms of strictly unital A_∞ -modules are already isomorphisms in the homotopy categories; no localization is needed. The main advantage is represented by the explicit structure of the morphisms in the derived categories themselves; no “roofs” manipulation is needed.

We discuss now the triangulated structures on the derived categories. The direct sum of two objects in $\mathbf{Mod}_\infty(A)$ is again a strictly unital A_∞ -module; the cohomological grading shift functor $\Sigma(M) := M[1]$ is actually an endofunctor on $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$. It follows that $\Sigma(\cdot) := \cdot[1]$ is an autoequivalence of $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$. More precisely, let (M, d_M) be an object of $\mathbf{D}^\infty(A)$. The bigraded object $M[1]$ can be endowed with a strictly unital A_∞ - A module structure as follows. The codifferential $d_{M[1]}$ has Taylor components $\bar{d}_{M[1]}^l : (M[1])[1] \otimes B[1]^{\otimes l} \rightarrow (M[1])[1]$ given by

$$\bar{d}_{M[1]}^l = -s \circ \bar{d}_M^l \circ (s^{-1} \otimes 1).$$

Proving that $d_{M[1]}^2 = 0$ is a straightforward sign-check. Given any morphism $F : M[1] \rightarrow N[1]$ in $\mathbf{D}^\infty(A)$ with Taylor components (of bidegree $(0,0)$) $\bar{F}^l : M[1] \otimes B[1]^{\otimes l} \rightarrow N[1]$, we get the induced morphism $\tilde{F} : (M[1])[1] \rightarrow (N[1])[1]$ in $\mathbf{Mod}_\infty(A)$ with Taylor components

$$\tilde{\bar{F}}^l = s \circ \bar{F}^l \circ (s^{-1} \otimes 1).$$

Once again, the proof of $\tilde{F} \circ d_{M[1]} = d_{N[1]} \circ \tilde{F}$ is a straightforward sign check. Same considerations hold in $\mathbf{D}^\infty(B)$. The inverse functor Σ^{-1} is given by $\Sigma^{-1}(\cdot) = \cdot[-1]$.

Definition 18 ([12]). *The triangulated structure on the derived category $\mathbf{D}^\infty(A)$ is given as follows. The autoequivalence Σ is simply the (cohomological) grading shift functor $\Sigma = [1]$. The distinguished triangles are isomorphic to those induced by semi-split sequences of strict A_∞ -morphisms*

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

in $\mathbf{Mod}_\infty A$, i.e. sequences such that

$$(22) \quad 0 \rightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \rightarrow 0$$

is an exact sequence in $\mathbf{bG}_\mathbb{K}$, and such that there exists a splitting $\rho \in \text{Hom}_{\mathbf{bG}_\mathbb{K}}(M', M)$ of f with

$$\rho \circ \bar{d}_M^i = \bar{d}_{M'}^i \circ (\rho \otimes 1^{\otimes i-1}), \quad i \geq 2.$$

³Augmentation w.r.t. a ground field \mathbb{K} of characteristic 0.

For the derived category of B the definition is analogous. The splitting ρ in the exact sequence (22) does not commute with the differentials \bar{d}_M^0 and $\bar{d}_{M'}^0$, in general. The above exact triangles endow $\mathbf{D}^\infty(A)$ with a triangulated category structure; the proof is contained in thm. 2.4.3.1 in [12]; the idea is induce the triangulated category structure on $\mathbf{D}^\infty(A)$ by using the one on $\mathbf{D}(UA)$, denoting by UA the enveloping algebra of A ; by definition UA is a differential (bi)graded algebra we refer to [12], [21] for all details; its derived category $\mathbf{D}(UA)$ is a well-known object. The equivalence of categories $\mathbf{D}(UA) \rightarrow \mathbf{D}^\infty(A)$ becomes then an equivalence of triangulated categories.

Let $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ be a distinguished triangle in $\mathbf{D}^\infty(A)$; it is isomorphic to a triangle of the form $M \xrightarrow{f} M' \xrightarrow{g} M'' \rightarrow M[1]$, with $M \xrightarrow{f} M' \xrightarrow{g} M''$ satisfying the hypothesis of the above definition. In more detail, let

$$0 \rightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \rightarrow 0$$

be a semi-split exact sequence with f, g strict, and splitting $\rho : M' \rightarrow M$, $\rho \circ f = 1$. This implies that

$$M_j^i \cong M_j^i \oplus M_j^i$$

as vector spaces over \mathbb{K} , for any $(i, j) \in \mathbb{Z}$; in virtue of this we assume that $M' = (M \oplus M'', d_{M \oplus M''})$, where $d_{M \oplus M''} = (d_M - h, d_{M''})$. It follows that $d_{M \oplus M''} \circ d_{M \oplus M''} = 0$ if and only if $h : M'' \rightarrow M[1]$ defines an A_∞ -morphism of strictly unital A_∞ - A -modules. Thanks to this, we will consider the semisplit exact sequence $0 \rightarrow M \xrightarrow{i} M' \xrightarrow{p} M'' \rightarrow 0$ with i and p the natural inclusion and projection (which are strict morphisms in $\mathbf{D}^\infty(A)$), and complete it to the exact triangle

$$(23) \quad M \xrightarrow{i} M' \xrightarrow{p} M'' \xrightarrow{h} M[1].$$

A small *memento*; in section 8.0.35 we will discuss the triangulated structure on some “deformed” derived categories of topologically free modules; some examples will be given: taking there the “limit” $\hbar = 0$ we obtain further examples of exact triangles in $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$.

6.0.19. *On the functors \mathcal{F} and \mathcal{G} .* Collecting the results on the derived categories of A and B and the definitions of the functors \mathcal{F} and \mathcal{G} we arrive at the pair of functors

$$\mathcal{F} : \mathbf{D}^\infty(A) \xrightarrow{F'} \mathbf{Mod}_\infty^{\text{strict}}(B) / \sim_{\text{strict}} \xrightarrow{i} \mathbf{D}^\infty(B),$$

and

$$\mathcal{G} : \mathbf{D}^\infty(B) \xrightarrow{G'} \mathbf{Mod}_\infty^{\text{strict}}(A) / \sim_{\text{strict}} \xrightarrow{i} \mathbf{D}^\infty(A),$$

with a little abuse of notation.

Proposition 12. *Let $(\mathcal{F}, \mathcal{G})$ be the pair of functors introduced above. Then $\mathcal{F}(A) \simeq K$, $\mathcal{F}(\overline{K}) \simeq B$, in $\mathbf{D}^\infty(B)$, and $\mathcal{G}(B) \simeq \underline{K}$, $\mathcal{G}(K) \simeq A$ in $\mathbf{D}^\infty(A)$.*

It follows that

$$\mathcal{F}(\mathcal{G}(K)) \simeq K \text{ in } \mathbf{D}^\infty(B), \quad \mathcal{G}(\mathcal{F}(A)) \simeq A \text{ in } \mathbf{D}^\infty(A).$$

Proof. The quasi-isomorphisms of strictly unital A_∞ -bimodules

$$K \rightarrow A \otimes_A K \rightarrow (K \otimes_B K) \otimes_A K = \mathcal{F}(\mathcal{G}(K))$$

and

$$A \rightarrow K \otimes_B K \rightarrow A \otimes_A (K \otimes_B K) \equiv (A \otimes_A K) \otimes_B K = \mathcal{G}(\mathcal{F}(A))$$

give both the statements. We used lem. 3, prop. 9 and prop. 11. \square

Lemma 9. *(\mathcal{F}, φ_1) and (\mathcal{G}, φ_2) are exact functors w.r.t the triangulated category structures on $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$; for any $M \in \mathbf{D}^\infty(A)$:*

$$\varphi_1(\mathcal{F}(M)) : (M \otimes_A K)[1] \rightarrow M[1] \otimes_A K, \quad \varphi_1(\mathcal{F}(M))(s(m, a_1 | \dots | a_l, k)) := (m | a_1 | \dots | a_l, k),$$

and similarly for φ_2 .

Proof. \mathcal{F} and \mathcal{G} send quasi-isomorphisms into quasi-isomorphisms as quasi-isomorphisms in the derived categories $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$ are homotopy equivalences. To prove that \mathcal{F} (and \mathcal{G}) are exact w.r.t. the triangulated structures on the derived categories it is sufficient to consider triangles of the form (23), i.e. $M \xrightarrow{i} M \oplus M' \xrightarrow{p} M' \xrightarrow{h} M[1]$.

Applying \mathcal{F} to such a triangle, and using the above lemmata we get the sequence

$$M \otimes_A \xrightarrow{\mathcal{F}(i)} (M \oplus M') \otimes_A K \xrightarrow{\mathcal{F}(p)} M' \otimes_A K \xrightarrow{\varphi_M(\mathcal{F}(M))} (M \otimes_A K)[1]$$

in $\mathbf{D}^\infty(B)$; the short exact sequence (\mathcal{F} is additive)

$$(24) \quad 0 \rightarrow M \underline{\otimes}_A K \xrightarrow{F(i)} (M \oplus M') \underline{\otimes}_A K \xrightarrow{F(p)} M'' \underline{\otimes}_A K \rightarrow 0$$

is semi-split w.r.t. the splitting

$$F(\rho) := \rho \otimes 1,$$

denoting by $\rho : M' \rightarrow M$ the splitting of the short exact sequence $0 \rightarrow M \xrightarrow{i} M \oplus M' \xrightarrow{p} M' \rightarrow 0$. In fact

$$F(\rho) \circ F(\alpha) = 1, \text{ and } F(\rho) \circ (s^{-1} \circ \bar{d}_{M' \underline{\otimes}_A K}^{0,i}) = (s^{-1} \circ \bar{d}_{M \underline{\otimes}_A K}^{0,i}) \circ (F(\rho) \otimes 1^{\otimes i}),$$

for $i \geq 1$. Then (24) can be completed to the distinguished triangle

$$M \underline{\otimes}_A K \xrightarrow{F(i)} (M \oplus M') \underline{\otimes}_A K \xrightarrow{F(p)} M'' \underline{\otimes}_A K \xrightarrow{h'} (M \underline{\otimes}_A K)[1],$$

with $h' := F(h)$. In summary \mathcal{F} sends exact triangles into exact triangles. Same considerations holds true for \mathcal{G} . \square

With $\mathbf{triang}_A^\infty(M)$ we denote the full triangulated subcategory in $\mathbf{D}^\infty(A)$ generated by $\{M[i]\langle j \rangle, i \in \mathbb{Z}\}$. $\mathbf{thick}_A^\infty(M_A)$, resp. $\mathbf{thick}_A^\infty(N_B)$ are the thick subcategories of direct summands of objects in $\mathbf{triang}_A^\infty(M_A)$, resp. $\mathbf{triang}_B^\infty(N_B)$. We refer to Appendix C for all definitions. Finally, we can state the main theorem of this section.

Theorem 7. *Let X be a finite dimensional vector space on $\mathbb{K} = \mathbb{R}$, or \mathbb{C} . Let (A, K, B) be the triple of A_∞ -structures with $A = S(X^*)$ and $B = \wedge(X)$ Koszul dual augmented differential bigraded algebras with zero differential and $K = \mathbb{K}$ endowed with the bigraded A_∞ - A - B -bimodule structure d_K given in [3]. The triangulated functor*

$$\mathcal{F} : \mathbf{D}^\infty(A) \rightarrow \mathbf{D}^\infty(B), \quad \mathcal{F}(\bullet) = \bullet \underline{\otimes}_A K$$

induces the equivalence of triangulated categories

$$\mathbf{triang}_A^\infty(A) \simeq \mathbf{triang}_B^\infty(K), \quad \mathbf{thick}_A^\infty(A) \simeq \mathbf{thick}_B^\infty(K).$$

Let $(\tilde{K}, d_{\tilde{K}})$ be the A_∞ - B - A -bimodule with $\tilde{K} = K$ and $d_{\tilde{K}}$ obtained from d_K exchanging A and B ; then the triangulated functor

$$\mathcal{F}'' : \mathbf{D}^\infty(B) \rightarrow \mathbf{D}^\infty(A), \quad \mathcal{F}''(\bullet) = \bullet \underline{\otimes}_B \tilde{K}$$

induces the equivalence of triangulated categories

$$\mathbf{triang}_A^\infty(\tilde{K}) \simeq \mathbf{triang}_B^\infty(B), \quad \mathbf{thick}_A^\infty(\tilde{K}) \simeq \mathbf{thick}_B^\infty(B).$$

Proof. Appendix B. \square

7. DEFORMATION QUANTIZATION OF A_∞ -STRUCTURES

In this section we study the quantizations $(A_\hbar, K_\hbar, B_\hbar)$ of the A_∞ -structures on the triple (A, K, B) . In this contest, the term “quantization”, or more properly, “Deformation Quantization” refers to a technique that produces new A_∞ -structures from already given A_∞ -data: the latter are recovered from the former through a “limiting” procedure. For the original idea we refer to [1]. A_∞ -structures on bigraded topologically free $\mathbb{K}[[\hbar]]$ -modules are said to be topological. The deformations are obtained through certain Feynman diagrams expansions, a “two branes” Formality theorem and an explicit choice of an \hbar -formal quadratic Poisson bivector $\pi_\hbar = \hbar\pi$ on X , the finite dimensional vector space underlying A and B . For the full construction and the 2-branes formality theorem we refer to [3]; the diagrammatic techniques there described generalize those introduced in [13]. The choice of a quadratic Poisson bivector field is motivated by the necessity of preserving the internal grading on the Deformation Quantization of triple (A, K, B) ; its main consequences are

- The Deformation Quantizations (A_\hbar, B_\hbar) of (A, B) are flat bigraded A_∞ -algebras.
- The Deformation Quantization K_\hbar of K is a left A_\hbar -module and a right B_\hbar -module with zero differential.
- It is possible to quantize the bimodules $A \underline{\otimes}_A K$, $K \underline{\otimes}_B B$, \underline{K} , $\underline{\text{End}}_A(K)$ and $\underline{\text{End}}_B(K)$ straightforwardly by using the “classical” A_∞ -bimodule structures with due changes.
- The quantized left and right derived actions are quasi-isomorphisms of topological A_∞ -algebras and topological A_∞ -bimodules.

7.0.20. *On modules over $\mathbb{K}[[\hbar]]$.* We consider the local ring $\mathbb{K}[[\hbar]]$ of formal power series with coefficients in \mathbb{K} . Topological free $\mathbb{K}[[\hbar]]$ modules are modules over $\mathbb{K}[[\hbar]]$ isomorphic to $\mathbb{K}[[\hbar]]$ -modules of the form $M[[\hbar]]$, with M a \mathbb{K} vector space. Let M and N be $\mathbb{K}[[\hbar]]$ -modules. The $\mathbb{K}[[\hbar]]$ -module $M \otimes_{\mathbb{K}[[\hbar]]} N$ is the quotient of the tensor product $M \otimes N$ ($\otimes = \otimes_{\mathbb{K}}$) by the subspace generated by all elements of the form $km \otimes n - m \otimes kn$, with $k \in \mathbb{K}[[\hbar]]$ and $m \in M$, $n \in N$. We denote by $\tilde{\otimes}$ the completed tensor product $M \tilde{\otimes} N$ of $M \otimes_{\mathbb{K}[[\hbar]]} N$. If M and N are topologically free, i.e. $M = M_1[[\hbar]]$ and $N = N_1[[\hbar]]$, then $M[[\hbar]] \tilde{\otimes} N[[\hbar]]$ is topologically free as well; in fact $M \tilde{\otimes} N = (M_1 \otimes N_1)[[\hbar]]$.

Let $\text{Hom}_{\mathbb{K}[[\hbar]]}(M[[\hbar]], N[[\hbar]])$ be the space of $\mathbb{K}[[\hbar]]$ -linear morphisms from $M[[\hbar]]$ to $N[[\hbar]]$; there exists an isomorphism $\mathcal{I} : \text{Hom}(M, N)[[\hbar]] \rightarrow \text{Hom}_{\mathbb{K}[[\hbar]]}(M[[\hbar]], N[[\hbar]])$ of $\mathbb{K}[[\hbar]]$ -modules. Any $\varphi \in \text{Hom}_{\mathbb{K}[[\hbar]]}(M[[\hbar]], N[[\hbar]])$ is uniquely determined by a formal power series

$$\sum_{i \geq 0} \varphi_i \hbar^i \in \text{Hom}(M, N)[[\hbar]].$$

We observe that any $\varphi \in \text{Hom}_{\mathbb{K}[[\hbar]]}(M[[\hbar]], N[[\hbar]])$ is continuous w.r.t. the \hbar -adic topology on $M[[\hbar]]$ and $N[[\hbar]]$. In the sequel we will use the formal power series description of morphisms extensively.

7.0.21. *On the category $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$.* Let $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ be the category of bigraded $\mathbb{K}[[\hbar]]$ -modules; an object in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ is a collection $\{M_j^i\}_{i,j \in \mathbb{Z}}$ of $\mathbb{K}[[\hbar]]$ -modules; the space of morphisms $\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[\hbar]]}}(M, N)$ is the object in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ with bihomogeneous components

$$\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[\hbar]]}}^{i,j}(M, N) = \prod_{r,s \in \mathbb{Z}} \text{Hom}_{\mathbb{K}[[\hbar]]}(M_s^r, N_{j+s}^{i+r}).$$

7.0.22. *Topologically free modules in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$.* We say that an object M_{\hbar} in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ is topologically free if

$$M_{\hbar} = \{(M_{\hbar})_j^i\}_{(i,j) \in \mathbb{Z}^2}, \text{ with } (M_{\hbar})_j^i = M_j^i[[\hbar]].$$

Let M_{\hbar} and N_{\hbar} be topologically free objects in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$, with $M_{\hbar} = M[[\hbar]]$ and $N_{\hbar} = N[[\hbar]]$, for M, N objects in $\mathbf{bG}_{\mathbb{K}}$; then $\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[\hbar]]}}(M_{\hbar}, N_{\hbar})$ is the topologically free object in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ with bihomogeneous components

$$\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[\hbar]]}}^{i,j}(M_{\hbar}, N_{\hbar}) = \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{i,j}(M, N)[[\hbar]].$$

For any topologically free M_{\hbar} in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$, the objects $M_{\hbar}[k]$ and $M_{\hbar}\langle l \rangle$ in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ are defined via

$$M_{\hbar}[k] = \{(M_{\hbar}[k])_j^i\}_{(i,j) \in \mathbb{Z}^2}, \quad (M_{\hbar}[k])_j^i := M_j^{i+k}[[\hbar]];$$

and

$$M_{\hbar}\langle l \rangle = \{(M_{\hbar}\langle l \rangle)_j^i\}_{(i,j) \in \mathbb{Z}^2}, \quad (M_{\hbar}\langle l \rangle)_j^i := M_{j+l}^i[[\hbar]];$$

for any $(k, l) \in \mathbb{Z}^2$. Topologically free objects in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ form a full subcategory in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ which is not abelian; we endow it with a monoidal structure induced by the completion $\tilde{\otimes}$, w.r.t the \hbar -adic topology, of the tensor product of topologically free $\mathbb{K}[[\hbar]]$ -modules. More precisely, for any M_{\hbar} and N_{\hbar} topologically free in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ and $(i, j) \in \mathbb{Z}^2$, we write (with a little abuse of notation)

$$(M_{\hbar} \tilde{\otimes} N_{\hbar})_j^i = \bigoplus_{\substack{i_1+i_2=i, \\ j_1+j_2=j}} M_{j_1}^{i_1}[[\hbar]] \tilde{\otimes} N_{j_2}^{i_2}[[\hbar]],$$

where $\tilde{\otimes}$ on the right hand side is the completed tensor product of topologically free $\mathbb{K}[[\hbar]]$ -modules introduced above.

8. TOPOLOGICAL A_{∞} -STRUCTURES

8.0.23. *Topological A_{∞} -algebras.*

Definition 19. Let A_{\hbar} be a topologically free object in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$. The topological tensor coalgebra over A_{\hbar} is the triple $(T(A_{\hbar}[1]), \Delta_{\hbar}, \epsilon_{\hbar})$ where

$$T(A_{\hbar}[1]) := \bigoplus_{q \geq 0} A_{\hbar}[1]^{\tilde{\otimes} q} = T(A[1])[[\hbar]]$$

in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$, and

$$\Delta_{\hbar} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}[[\hbar]]}}^{0,0}(T(A_{\hbar}[1]), T(A_{\hbar}[1]) \tilde{\otimes} T(A_{\hbar}[1]))$$

given by $\Delta_{\hbar} = \sum_{i \geq 0} \Delta^{(i)} \hbar^i = \Delta^{(0)} = \Delta$, where Δ denotes the coproduct on $T(A[1])$ and $\epsilon_{\hbar} = \epsilon$, where ϵ is the counit in $T(A[1])$.

By definition $(1 \tilde{\otimes} \Delta_{\hbar}) \circ \Delta_{\hbar} = (\Delta_{\hbar} \tilde{\otimes} 1) \circ \Delta_{\hbar}$ and $(\epsilon_{\hbar} \tilde{\otimes} 1) \circ \Delta_{\hbar} = (1 \tilde{\otimes} \epsilon_{\hbar}) \circ \Delta_{\hbar} = 1$.

8.0.24. *On codifferentials: definitions.*

Definition 20. A coderivation on $T(A_h[1])$ is a morphism $d_{A_h} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}[[h]]}^{1,0}(T(A_h[1]), T(A_h[1]))$ s. t. $(1 \tilde{\otimes} d_{A_h} + d_{A_h} \tilde{\otimes} 1) \circ \Delta_h = \Delta_h \circ d_{A_h}$ and

$$(25) \quad d_{A_h}^2 = 0$$

Let d_{A_h} be the coderivation on $T(A_h[1])$ uniquely determined by the formal power series

$$d_{A_h} = \sum_{i \geq 0} d_{A_h}^{(i)} h^i, \quad d_{A_h}^{(i)} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{1,0}(T(A[1]), T(A[1])).$$

Then, by definition of d_{A_h} , each $d_{A_h}^{(i)}$ is uniquely determined by the family of Taylor components $d_{A_h}^{(i),k} = p_{A[1]} \circ d_{A_h}^{(i)}|_{A[1]^{\otimes k}}$. The quadratic relations (25) are equivalent to a tower of quadratic relations with the Taylor components $d_{A_h}^{k,(i)}$, $k \geq 0$, $i \geq 0$.

Definition 21. Let A_h be a topologically free object in $\mathbf{bG}_{\mathbb{K}}[[h]]$. A topological A_∞ -algebra structure on A_h is the datum of a coderivation on the topological tensor coalgebra over A_h .

Lemma 10. Let (A_h, d_{A_h}) be a topological A_∞ -algebra. Then (A, d_A) , $d_A := d_{A_h}^{(0)}$, is an A_∞ -algebra (on \mathbb{K}).

8.0.25. *Topologically free A_∞ -modules.*

Definition 22. Let M_h be topologically free module in $\mathbf{bG}_{\mathbb{K}}[[h]]$; $\mathcal{R}_h(M_h)$ is the object

$$\mathcal{R}_h(M_h) := M_h \tilde{\otimes} T(A_h[1]) = (M[1] \otimes T(A[1]))[[h]]$$

in $\mathbf{bG}_{\mathbb{K}}[[h]]$. A right $(T(A_h[1]), \Delta_h, \epsilon_h)$ -counital-comodule structure on $\mathcal{R}_h(M_h)$ is the morphism

$$\Delta_h^R \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}[[h]]}^{0,0}(\mathcal{R}_h(M_h), \mathcal{R}_h(M_h) \tilde{\otimes} T(A_h[1])), \quad \Delta_h^R = \Delta_h^{R,(0)} = \Delta^R,$$

satisfying $(1 \tilde{\otimes} \Delta_h) \circ \Delta_h^R = (\Delta_h^R \tilde{\otimes} 1) \circ \Delta_h^R$ and $(1 \tilde{\otimes} \epsilon_h) \circ \Delta_h^R = 1$, denoting by Δ^R the usual counital- $T(A[1])$ -comodule structure on $M[1] \otimes T(A[1])$.

Definition 23. A codifferential on the right $T(A_h[1])$ -comodule $\mathcal{R}_h(M_h)$ is a morphism $d_{M_h} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}[[h]]}^{1,0}(\mathcal{R}_h(M_h), \mathcal{R}_h(M_h))$ s.t. $\Delta_h^R \circ d_{M_h} = (1 \tilde{\otimes} d_{M_h} + d_{A_h} \tilde{\otimes} 1) \circ \Delta_h^R$ and

$$(26) \quad d_{M_h}^2 = 0.$$

By definition, if $d_{M_h} = \sum_{i \geq 0} d_{M_h}^{(i)} h^i$, then each $d_{M_h}^{(i)} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{1,0}(M[1] \otimes T(A[1]), M[1] \otimes T(A[1]))$ is uniquely determined by its Taylor components $d_{M_h}^{(i),n} = p_{M[1]} \circ d_{M_h}^{(i)}|_{M[1] \otimes A[1]^{\otimes n}}$, for any $i, n \geq 0$. The quadratic relations (26) are equivalent to a tower of quadratic relations involving the aforementioned maps $d_{M_h}^{(i),n}$.

Definition 24. Let M_h be an object in $\mathbf{bG}_{\mathbb{K}}[[h]]$. A topological right A_∞ - A_h -module structure on M_h is the datum of a codifferential d_{M_h} on $\mathcal{R}_h(M_h)$.

Lemma 11. Let M_h be a topological right A_∞ - A_h -module. Then M is a right A_∞ - A -module.

In the same spirit, one can define topological left A_∞ -modules and topological A_∞ -bimodules, with due changes.

8.0.26. *On morphisms, quasi-isomorphisms and homotopy equivalences.*

Definition 25. Let (M_h, d_{M_h}) and (N_h, d_{N_h}) be topological A_∞ - A_h -modules, with (A_h, d_{A_h}) topological A_∞ -algebra.

A morphism $f_h : M_h \rightarrow N_h$ of topological A_∞ - A_h -modules is a map $f_h \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}[[h]]}^{0,0}(\mathcal{R}_h(M_h), \mathcal{R}_h(N_h))$ which is a morphism of $T(A_h[1])$ -counital-comodules s.t.

$$d_{N_h} \circ f_h = f_h \circ d_{M_h}.$$

Such a morphism is uniquely determined by a formal power series $f_h = \sum_{i \geq 0} f_h^{(i)} h^i$, with $f_h^{(i)} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{0,0}(M[1] \otimes T(A[1]), N[1] \otimes T(A[1]))$, for any $i \geq 0$. Each component $f_h^{(i)}$ is a morphism of counital- $T(A[1])$ -comodules, and so it admits an explicit description by Taylor components $f_h^{(i),n} : M[1] \otimes A[1]^{\otimes n} \rightarrow N[1]$, for any $i, n \geq 0$.

Lemma 12. Let $f_h : M_h \rightarrow N_h$, $f_h = \sum_{i \geq 0} f_h^{(i)} h^i$ be a morphism of topological A_∞ - A_h -modules. Then $f_h^{(0)} : M \rightarrow N$ is a morphism of A_∞ - A -modules.

Definition 26. Let $f_{\hbar}, g_{\hbar} : M_{\hbar} \rightarrow N_{\hbar}$ be morphisms of topological A_{∞} - A_{\hbar} -modules; we say that they are topological A_{∞} -homotopy equivalent (alternatively: top. A_{∞} -homotopic) if there exists a topological A_{∞} -homotopy between them, i.e. a map $H_{\hbar} : M_{\hbar} \rightarrow N_{\hbar}$ of $T(A_{\hbar}[1])$ -counital-comodules with

$$H_{\hbar} = \sum_{i \geq 0} H_{\hbar}^{(i)} \hbar^i,$$

$$H_{\hbar}^{(i),n} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{-1,0}(M[1] \otimes A[1]^{\otimes n}, M[1]), \quad n \geq 0,$$

such that

$$f_{\hbar} - g_{\hbar} = d_{N_{\hbar}} \circ H_{\hbar} + H_{\hbar} \circ d_{M_{\hbar}}$$

holds true, order by order in \hbar .

8.0.27. *On units.* Let A_{\hbar} in $\mathbf{bG}_{\mathbb{K}}[[\hbar]]$ be a topological A_{∞} -algebra with codifferential $d_{A_{\hbar}}$. We say that the right A_{∞} - A_{\hbar} -module structure $d_{M_{\hbar}}$ on M_{\hbar} is strictly unital if

$$d_{M_{\hbar}}^{(i),n}(m|a_1| \dots |\eta| \dots |a_n) = 0,$$

for any $n \geq 2$ and $i \geq 0$. We have denoted by η the unit in A and by $d_{M_{\hbar}}^n$ the n -th Taylor component of $d_{M_{\hbar}}^{(i)}$.

A morphism $f_{\hbar} : M_{\hbar} \rightarrow N_{\hbar}$ of topological A_{∞} - A_{\hbar} -modules is strictly unital if

$$f_{\hbar}^{(i),n}(m|a_1| \dots |\eta| \dots |a_n) = 0,$$

for any $n \geq 1$, $i \geq 0$, where $f_{\hbar}^{(i),n}$ is the n -th Taylor component of $f_{\hbar}^{(i)}$.

Strictly unital homotopies are defined similarly.

8.0.28. *Quantizing (A, K, B) via quadratic Poisson structures.* By X we denote a finite dimensional vector space of dimension n on $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $(T_{poly}(X)[[\hbar]], [\cdot, \cdot]_{\hbar})$ be the trivial deformation of $(T_{poly}(X), [\cdot, \cdot])$, with Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\hbar}$ obtained by extending $[\cdot, \cdot]$ $\mathbb{K}[[\hbar]]$ -linearly. Let $\{x_i\}_{i \in I}$ be a set of global coordinates on X , with $\sharp I = n$. We say that the Poisson bivector $\pi \in T_{poly}(X)$ is quadratic if it can be written as

$$\pi = \sum_{i,j=1}^n \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \pi^{ij} = \sum_{k,l=1}^n c_{kl}^{ij} x_k x_l,$$

for some constant coefficients $c_{kl}^{ij} \in \mathbb{K}$ such that $c_{kl}^{ij} = -c_{kl}^{ji}$, for any $k, l \in I$. In [3] a 2-branes Formality theorem is proved; we refer to [3] for all details; here we sketch the construction in the special case in which the triple (A, K, B) appears.

To the A_{∞} -triple (A, K, B) it is possible to associate a unital A_{∞} -category $\text{Cat}_{\infty}(A, K, B)$; its objects are the branes $U = X$ and $V = \{0\}$ in the vector space X and the spaces of morphisms are given by $\text{Hom}(U, U) = A$, $\text{Hom}(V, V) = B$, $\text{Hom}(U, V) = K$ and $\text{Hom}(V, U) = 0$. In this “local” setting the branes are linear subspaces on the ambient space X . The unital A_{∞} -category structure on $\text{Cat}_{\infty}(A, K, B)$ is induced by the associative algebra structures on A , B and the A_{∞} - A - B -bimodule structure on K . The 2-branes Formality theorem states the existence of a quasi-isomorphism of L_{∞} -algebras

$$\mathcal{U} : (T_{poly}^{\bullet+1}(X), [\cdot, \cdot], 0) \rightarrow (C^{\bullet+1}(\text{Cat}_{\infty}(A, K, B)), [\cdot, \cdot]_G, \partial)$$

between the differential graded Lie algebra (shortly, DGLA) of polynomial polyvector fields on X and the DGLA of Hochschild cochains on the A_{∞} -category $\text{Cat}_{\infty}(A, K, B)$ endowed with the Gerstenhaber bracket $[\cdot, \cdot]_G$ and Hochschild differential ∂ . As a graded object $C^{\bullet+1}(\text{Cat}_{\infty}(A, K, B))$ decomposes in the direct sum of three components: $C^{\bullet+1}(A, A)$, $C^{\bullet+1}(B, B)$ and $C^{\bullet+1}(A, K, B)$. $C^{\bullet+1}(A, A)$ and $C^{\bullet+1}(B, B)$ are the DGLAs of Hochschild cochains of A and B ; they are sub complexes of $C^{\bullet+1}(\text{Cat}_{\infty}(A, K, B))$. $C^{\bullet+1}(A, K, B)$ is given by

$$C^n(A, K, B) = \bigoplus_{p+q+r=n-1} \text{Hom}^q(A^{\otimes p} \otimes K \otimes B^{\otimes r}, K).$$

The proof of the 2-branes Formality theorem is based on Stokes’ theorem on manifolds with corners and the properties of the 4-color propagators ([5],[3],[17]) at the boundary components. In the general case, we have to consider short loops in the Feynman diagrams describing the L_{∞} -quasi-isomorphism \mathcal{U} .

The L_{∞} -quasi-isomorphism \mathcal{U} induces an isomorphism between the sets of Maurer-Cartan elements (MCEs) on the DGLAs $(T_{poly}^{\bullet+1}(X), [\cdot, \cdot], 0)$ and $(C^{\bullet+1}(\text{Cat}_{\infty}(A, K, B)), [\cdot, \cdot]_G, \partial)$. MCEs in $T_{poly}(X)$ are Poisson structures on X ; they are mapped to MCEs on $C^{\bullet+1}(A, A)$ and $C^{\bullet+1}(B, B)$ which are A_{∞} -deformations of the graded associative algebra structures on A and B and to an A_{∞} -deformation of the A_{∞} - A - B -bimodule structure on K .

Let $\hbar\pi$ be a MCE in $T_{poly}(X)[[\hbar]]$; it satisfies

$$[\hbar\pi, \hbar\pi]_{\hbar} = 0$$

order by order in \hbar , denoting by $[\cdot, \cdot]_\hbar$ the Lie bracket on $T_{poly}(X)[[\hbar]]$ obtained by extending $\mathbb{K}[[\hbar]]$ -linearly the Lie bracket $[\cdot, \cdot]$ on $T_{poly}(X)$.

Let $\mathcal{U}(\hbar\pi) = \mathcal{U}_A(\hbar\pi) + \mathcal{U}_B(\hbar\pi) + \mathcal{U}_K(\hbar\pi)$ be the MCE in $C^{\bullet+1}(\text{Cat}_\infty(A, K, B))[[\hbar]]$ where $\mathcal{U}_A(\hbar\pi)$ is the component of \mathcal{U} on $C^{\bullet+1}(A, A)[[\hbar]]$, $\mathcal{U}_B(\hbar\pi)$ is the one on $C^{\bullet+1}(B, B)[[\hbar]]$ and $\mathcal{U}_K(\hbar\pi)$ on $C^{\bullet+1}(A, K, B)[[\hbar]]$. The three components are defined through an expansion in Feynman graphs in which “areal vertices” appear.

Proposition 13 ([3], section 8.1). *Let $\hbar\pi$ be an \hbar -formal quadratic MCE in $T_{poly}(X)[[\hbar]]$.*

- *The A_∞ -deformations of A , resp. B are given by $(A[[\hbar]], \cdot + \mathcal{U}_A(\hbar\pi))$, $(B[[\hbar]], \wedge + \mathcal{U}_B(\hbar\pi))$.
In other words, A and B are deformed into bigraded associative algebras with zero differential. The deformed products preserves the internal grading.*
- *The A_∞ - $A[[\hbar]]$ - $B[[\hbar]]$ -bimodule deforming K is given by $(K[[\hbar]], d_{K_\hbar})$, $d_{K_\hbar} = d_K + \mathcal{U}_K(\hbar\pi)$.
The codifferential d_{K_\hbar} is such that*

$$d_{K_\hbar}^{(i),n,0} = d_{K_\hbar}^{(i),0,m} = 0$$

if either $m = n = 0$ or $m, n, \geq 2$, for any $i \geq 0$.

Choosing a general Poisson structure π on X we obtain different quantizations A_\hbar resp. B_\hbar of A , resp. B , in general curved as A_∞ -algebras. For a curved example we refer to [2].

8.0.29. *Quantizing bimodules.* In section 2 we have defined left and right bar resolutions of A_∞ -bimodules.

The Taylor components of the codifferential on such resolutions are given by the formulæ (6). We quantize the resolutions considering the triple $(A_\hbar, K_\hbar, B_\hbar)$.

Definition-lemma 1. *Let (M_\hbar, d_{M_\hbar}) be a topological A_∞ - A_\hbar - B_\hbar bimodule. The left topological bar resolution of M_\hbar is the object $(A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar = (A_\hbar \tilde{\otimes} T(A_\hbar[1]) \tilde{\otimes} M_\hbar)$ in $\mathbf{bG}_{\mathbb{K}}[[\hbar]]$. It is a topological A_∞ - A_\hbar - B_\hbar -bimodule with codifferential $d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar} = \sum_{i \geq 0} d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar}^{(i)} \hbar^i$. For any $i \geq 0$, the (k, l) -th Taylor component*

$$d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar}^{(i),k,l} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{1,0}((A[1]^{\otimes k} \otimes (A \underline{\otimes}_A M)[1]) \otimes B[1]^{\otimes l}, (A \underline{\otimes}_A M)[1])$$

of $d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar}^{(i)}$ is given by the formulæ (6) with the insertion of the operators $d_{A_\hbar}^{(i),2}$, $d_{M_\hbar}^{(i),\cdot,\cdot}$ and $d_{B_\hbar}^{(i),2}$.

Proof. It is easy but quite long to check that $d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar} \circ d_{A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar} = 0$ follows from associativity of the products on A_\hbar , B_\hbar and the quadratic relations $d_{M_\hbar}^2 = 0$. \square

We can define right topological bar resolutions, or bar resolutions of topological A_∞ -bimodules, with due changes. In the sequel we will consider the bimodule K_\hbar and the topological bar resolutions $A_\hbar \tilde{\otimes}_{A_\hbar} K_\hbar$ and $K_\hbar \tilde{\otimes}_{B_\hbar} B_\hbar$.

Let \underline{K} and \overline{K} be the A_∞ - B - A -bimodules introduced in section 6.

Definition-lemma 2. *The quantization \underline{K}_\hbar of the A_∞ - B - A -bimodule \underline{K} is the object*

$$\underline{K}_\hbar = \mathbf{Hom}_{B_\hbar}(K_\hbar \tilde{\otimes}_{B_\hbar} B_\hbar, B_\hbar)$$

in $\mathbf{bG}_{\mathbb{K}}[[\hbar]]$. It is a strictly unital topological A_∞ - B_\hbar - A_\hbar -bimodule with codifferential $d_{\underline{K}_\hbar} = \sum_{i \geq 0} d_{\underline{K}_\hbar}^{(i)} \hbar^i$. For any $i \geq 0$, the (k, l) -th Taylor component

$$d_{\underline{K}_\hbar}^{(i),k,l} \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{1,0}((A[1]^{\otimes k} \otimes \underline{K}[1] \otimes B[1]^{\otimes l}, \underline{K}[1])$$

of $d_{\underline{K}_\hbar}^{(i)}$ is given by the formulæ in lemma 5, with the insertion of the operators $d_{A_\hbar}^{(i),2}$, $d_{K_\hbar}^{(i),\cdot,\cdot}$ and $d_{B_\hbar}^{(i),2}$.

Proof. The proof of the topological A_∞ -bimodule structure is similar to the one for $A_\hbar \tilde{\otimes}_{A_\hbar} M_\hbar$; we use the associativity of the products on A_\hbar , B_\hbar and the topological A_∞ -bimodule structure on K_\hbar . \square

The definition of \overline{K}_\hbar is analogous, with due changes. Similarly, we can introduce the quantizations $\underline{\text{End}}_{B_\hbar}(K_\hbar)$, resp. $\underline{\text{End}}_{A_\hbar}(K_\hbar)$ of $\underline{\text{End}}_B(K)$, resp. $\underline{\text{End}}_A(K)$; their topological A_∞ -algebra structures are induced by the topological A_∞ -structures on $(A_\hbar, K_\hbar, B_\hbar)$. We use the same classical formulæ introduced in section 6, with due changes. Such quantizations are topologically free objects in $\mathbf{bG}_{\mathbb{K}}[[\hbar]]$. Let $\mathcal{B}_{A_\hbar}(K_\hbar) = A_\hbar \tilde{\otimes}_{A_\hbar} K_\hbar$, $\mathcal{B}_{B_\hbar}(K_\hbar) = K_\hbar \tilde{\otimes}_{B_\hbar} B_\hbar$ and $\mathbf{End}_{A_\hbar}(\mathcal{B}_{A_\hbar}(K_\hbar))$, $\mathbf{End}_{B_\hbar}(\mathcal{B}_{B_\hbar}(K_\hbar))$ be the topologically free objects in $\mathbf{bG}_{\mathbb{K}}[[\hbar]]$

$$\mathbf{End}_{B_\hbar}(\mathcal{B}_{B_\hbar}(K_\hbar)) = \{\varphi \in \text{End}_{\mathbf{bG}_{\mathbb{K}}[[\hbar]]}(\mathcal{B}_{B_\hbar}(K_\hbar)), B_\hbar - \text{linear}\},$$

$$\mathbf{End}_{A_\hbar}(\mathcal{B}_{A_\hbar}(K_\hbar)) = \{\varphi \in \text{End}_{\mathbf{bG}_{\mathbb{K}}[[\hbar]]}(\mathcal{B}_{A_\hbar}(K_\hbar)), A_\hbar - \text{linear}\}$$

They are canonically endowed with topological A_∞ -bimodule structures; the formulæ are induced by the classical constructions presented in the previous sections. The constructions used to quantize $\underline{\text{End}}_A(K)$ and $\underline{\text{End}}_B(K)$ are replied here, with due changes.

Proposition 14. *The quantized derived actions*

$$\begin{aligned} L_{A_h} : A_h &\rightarrow \underline{\text{End}}_{B_h}(K_h), \\ R_{B_h} : B_h &\rightarrow \underline{\text{End}}_{A_h}(K_h)^{op}, \end{aligned}$$

are quasi-isomorphisms of topological A_∞ -algebras.

Proof. The proposition is proved in [3], section 8.1. □

Corollary 5. L_{A_h} and R_{B_h} descend to quasi-isomorphisms of topological A_∞ -bimodules.

Proof. The bimodule structures on A_h , B_h , $\underline{\text{End}}_{B_h}(K_h)$ and $\underline{\text{End}}_{A_h}(K_h)$ are those described in section 6 for A , B , $\underline{\text{End}}_B(K)$ and $\underline{\text{End}}_A(K)$ with due changes. □

8.0.30. *Some quasi-isomorphisms of quantized bimodules.*

Proposition 15. • *There exist quasi-isomorphisms*

$$\begin{aligned} \mu_{K_h} : K_h \tilde{\otimes}_{B_h} B_h &\rightarrow K_h, & \mu'_{K_h} : A_h \tilde{\otimes}_{A_h} K_h &\rightarrow K_h, \\ \Phi_{K_h} : K_h &\rightarrow K_h \tilde{\otimes}_{B_h} B_h, & \Phi'_{K_h} : K_h &\rightarrow A_h \tilde{\otimes}_{A_h} K_h. \end{aligned}$$

of strictly unital topological A_∞ - A_h - B_h -bimodules.

- *There exists an isomorphism*

$$\Theta_h^1 : K_h \tilde{\otimes}_{B_h} \underline{K}_h \rightarrow \mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h)),$$

of strictly unital topological A_∞ - A_h - A_h -bimodules.

- *There exists an isomorphism*

$$\Theta_h^2 : \overline{K}_h \tilde{\otimes}_{A_h} K_h \rightarrow \mathbf{End}_{A_h}(\mathcal{B}_{A_h}(K_h))^{op}$$

of strictly unital topological A_∞ - B_h - B_h -bimodules.

Proof. On μ_{K_h} . The morphism is defined using the formulæ for the morphism μ in proposition 2, section 2, with due changes. So the compatibility with the topological A_∞ -bimodule structures follows. In other words,

$$\mu_{K_h} \left(\sum_{i \geq 0} a_i h^i \right) = \sum_{n \geq 0} \sum_{i+j=n} \mu_1^{(i)}(a_j) h^n,$$

with $\mu_1^{(i)} \in \text{Hom}_{\mathbf{bG}_K}^{0,0}(\mathcal{T}(A[1]) \otimes (A \underline{\otimes}_A K)[1] \otimes \mathcal{T}(B[1]), K[1])$ uniquely determined by the Taylor components $\mu_1^{(i),k,l}$, with

$$\mu_1^{(i),k,l}(a_1 | \dots | a_k, s(1|b_1 | \dots | b_q | b), b'_1 | \dots | b'_l) = \pm d_{K_h}^{(i),k,q+1+l}(a_1 | \dots | a_k | 1|b_1 | \dots | b_q | b | b'_1 | \dots | b'_l),$$

The relations $d_{K_h} \circ \mu_{K_h} = \mu_{K_h} \circ d_{K_h \tilde{\otimes}_{B_h} B_h}$ are equivalent to $\sum_{i+j=n} d_{K_h}^{(j)} \circ d_{K_h}^{(i)} = 0$, for any $n \geq 0$. We recall that in the classical case the relations

$$d_K \circ \mu = \mu \circ d_{K \underline{\otimes}_B B}$$

are equivalent to $d_K^2 = 0$.

It is also clear that $\mu_{K_h}|_{h=0} = \mu_{K_h}|_{h=0} = \mu$; as μ is a quasi-isomorphism, then μ_{K_h} is a quasi-isomorphism as well. Similar considerations hold for μ'_{K_h} . For Φ_{K_h} and Φ'_{K_h} the conclusions are similar, with due changes: all we need is to consider the trivially “quantized” version of the formulæ introduced in lemma 3. Θ_h^1 is given by the formal power series $\Theta_h^1 = \sum_{i \geq 0} \Theta_h^{1,(i)} h^i$, with $\Theta_h^{1,(i)} = 0$ for $i \geq 1$ and $\Theta_h^{1,(0)} = \mathcal{I} \circ G$, where $\mathcal{I} : K \underline{\otimes}_B \underline{K} \rightarrow \mathcal{B}_B(K) \otimes \underline{K}$ is an isomorphism of A_∞ - A - A -bimodules and $G : \mathcal{B}_B(K) \otimes \underline{K} \rightarrow \mathbf{End}_A(\mathcal{B}_B(K))$ is the isomorphism A_∞ - A - A -bimodules given in prop. 8.

On any element $\sum_{i \geq 0} v_i h^i$ in $(\mathcal{T}(A[1]) \otimes (K \underline{\otimes}_B \underline{K})[1] \otimes \mathcal{T}(A[1]))[[h]]$ the relations

$$d_{\mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h))} \circ \Theta_h^1 = \Theta_h^1 \circ d_{K_h \tilde{\otimes}_{B_h} \underline{K}_h}$$

are equivalent to

$$\sum_{i+j=n} d_{\mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h))}^{(j)} (\Theta_h^{1,(0)}(v_i)) = \sum_{i+j=n} \Theta_h^{1,(0)}(d_{K_h \tilde{\otimes}_{B_h} \underline{K}_h}^{(j)}(v_i))$$

for any $n \geq 0$; the above relations are verified (projecting both sides onto $\mathbf{End}_{A_h}(\mathcal{B}_{B_h}(K_h))$, as usual) as $\Theta_h^{1,(0)}$ commutes with $d_{\mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h))}^{(j)}$, $d_{K_h \tilde{\otimes}_{B_h} K_h}^{(j)}$, for any $j \geq 0$.

We recall that the Taylor components $d_{\mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h))}^{(j),\cdot,\cdot}$ and $d_{K_h \tilde{\otimes}_{B_h} K_h}^{(j),\cdot,\cdot}$ are given by the Taylor components $d_{\mathbf{End}_B(\mathcal{B}_B(K))}^{(j),\cdot,\cdot}$ and $d_{K \otimes_B K}^{(j),\cdot,\cdot}$ via the substitutions $d_A^2 \mapsto d_{A_h}^{(j),2}$, $d_B^2 \mapsto d_{B_h}^{(j),2}$ and $d_{K_h}^{(j),\cdot,\cdot} \mapsto d_{K_h}^{(j),\cdot,\cdot}$.

$\Theta_h^{1,(0)}$ is an isomorphism in $\mathbf{bG}_{\mathbb{K}}$; it follows that Θ_h^1 is an isomorphism in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$. For Θ_h^2 similar considerations hold, with due changes. \square

Corollary 6. • *There exists a homotopy equivalence*

$$K_h \tilde{\otimes}_{A_h} K_h \rightarrow \underline{\mathbf{End}}_{B_h}(K_h),$$

of strictly unital topological A_∞ - A_h - A_h -modules.

• *There exists a homotopy equivalence*

$$\overline{K}_h \tilde{\otimes}_{A_h} K_h \rightarrow \underline{\mathbf{End}}_{A_h}(K_h)^{op}$$

of strictly unital topological A_∞ - B_h - B_h -modules.

Proof. The classical homotopy equivalence $H : \mathbf{End}_B(\mathcal{B}_B(K)) \rightarrow \underline{\mathbf{End}}_B(K)$ defined in prop. 7 induces the homotopy equivalence $H_h : \mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h)) \rightarrow \underline{\mathbf{End}}_{B_h}(K_h)$, $H_h = H$. The check is immediate; in fact the codifferentials on $\mathbf{End}_{B_h}(\mathcal{B}_{B_h}(K_h))$ and $\underline{\mathbf{End}}_{B_h}(K_h)$ are constructed by using $d_{K_h}^{(i),k,l}$, $d_{A_h}^{(i),j}$ and $d_{B_h}^{(i),j}$. It is necessary to prove the compatibility of the quantized homotopy equivalence with these operators. But this goes on like in the classical case. Similar considerations hold for the second statement, with due changes. \square

Composing with the quantized derived action we arrive at

Corollary 7. • *There exists a quasi-isomorphism*

$$A \rightarrow K_h \tilde{\otimes}_{A_h} K_h,$$

of strictly unital topological A_∞ - A_h - A_h -modules.

• *There exists a quasi-isomorphism*

$$B_h \rightarrow \overline{K}_h \tilde{\otimes}_{A_h} K_h.$$

of strictly unital topological A_∞ - B_h - B_h -modules.

8.0.31. *On the categories $\mathbf{Mod}_{tf}^\infty(A_h)$ and $\mathbf{Mod}_{tf}^\infty(B_h)$.*

Definition 27. *Let (A_h, K_h, B_h) be the triple quantizing (A, K, B) w.r.t. an \hbar -formal quadratic Poisson bivector π_h .*

- $\mathbf{Mod}_{tf}^\infty(A_h)$ *is the category of strictly unital topological A_∞ -right- A_h -modules.*
- $\mathbf{Mod}_{tf}^\infty(B_h)$ *is the category of strictly unital topological A_∞ -right- B_h -modules.*

$\mathbf{Mod}_{tf}^\infty(A_h)$ and $\mathbf{Mod}_{tf}^\infty(B_h)$ are additive categories. The direct sum $M_h \tilde{\oplus} N_h$ of objects (M_h, d_{M_h}) and (N_h, d_{N_h}) in $\mathbf{Mod}_{tf}^\infty(A_h)$ (or $\mathbf{Mod}_{tf}^\infty(B_h)$) is the topologically free module

$$M_h \tilde{\oplus} N_h := (M \oplus N) [[\hbar]]$$

if $M_h = M [[\hbar]]$ and $N_h = N [[\hbar]]$ in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$, endowed with the strictly unital topological A_∞ -module structure given by the codifferential $d_{M_h \tilde{\oplus} N_h}$. The natural inclusion and projection

$$i_h : M_h \rightarrow M_h \tilde{\oplus} N_h, \quad p_h : M_h \tilde{\oplus} N_h \rightarrow N_h,$$

are the strict topological A_h -module morphisms defined *via*

$$i_h = i_h^{(0)}, \quad i_h^{(0)} = i_h^{(0),0} = i : M \rightarrow M \oplus N, \quad m \mapsto m \oplus 0,$$

and

$$p_h = p_h^{(0)}, \quad p_h^{(0)} = p_h^{(0),0} = p : M \oplus N \rightarrow N, \quad m \oplus n \mapsto n.$$

8.0.32. *On quasi-isomorphisms in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ and $\mathbf{Mod}_{tf}^\infty(B_\hbar)$.* The category $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ of all bigraded $\mathbb{K}[[\hbar]]$ -modules is abelian; clearly $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ and $\mathbf{Mod}_{tf}^\infty(B_\hbar)$ are (not full) subcategories of $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$.

In general, the cohomology of a topologically free differential bigraded $\mathbb{K}[[\hbar]]$ -module is not topologically free; so we introduce the following definition.

Definition 28. *A quasi-isomorphism $f_\hbar : N_\hbar \rightarrow M_\hbar$ of objects in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ (or $\mathbf{Mod}_{tf}^\infty(B_\hbar)$) is a morphism of topological A_∞ -modules s. t. $H(f_\hbar) : H(N_\hbar) \rightarrow H(M_\hbar)$ is an isomorphism in the abelian category $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$.*

Quasi-isomorphisms of strictly unital top. A_∞ -modules are not, in general, homotopy equivalences. A counterexample is given by

Example 1 (B. Keller, [8]). *Let $A_\hbar = \mathbb{K}[[\hbar]]$ and (M_\hbar, d_{M_\hbar}) be the object in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ given by*

$$M_\hbar = \{M_0^0[[\hbar]], M_0^1[[\hbar]], M_0^2[[\hbar]]\}, \quad M_0^0 = M_0^1 = \mathbb{K}, \quad M_0^2 = 0,$$

with codifferential d_{M_\hbar} s.t. $d_{M_\hbar}^{(0),0} : M_0^0[[\hbar]] \rightarrow M_0^1[[\hbar]]$ is the multiplication by \hbar , and $d_{M_\hbar}^{(0),1} : M_0^1\tilde{\otimes}A_\hbar \rightarrow M_0^2$ is the multiplication in $\mathbb{K}[[\hbar]]$. All the other components are set to be zero. The strict quasi-isomorphism (\mathbb{K} is concentrated in bidegree $(0,0)$) of strictly unital top. A_∞ - A_\hbar -modules

$$f_\hbar : M_\hbar \rightarrow \mathbb{K}$$

admits no inverse g_\hbar (up to homotopy); in fact such an inverse would have a $\mathbb{K}[[\hbar]]$ -linear $(0,0)$ -th component $g_\hbar^{0,0} : \mathbb{K} \rightarrow \mathbb{K}[[\hbar]]$. But this implies that g_\hbar is the zero map.

8.0.33. *On $\mathcal{H}_\infty^{tf}(A_\hbar)$, $\mathcal{H}_\infty^{tf}(B_\hbar)$ and their triangulated structures.* As $\mathbf{Mod}_{tf}^\infty(A_\hbar)$ and $\mathbf{Mod}_{tf}^\infty(B_\hbar)$ are additive categories, we can introduce the homotopy categories

$$\mathcal{H}_\infty^{tf}(A_\hbar) := \mathcal{H}(\mathbf{Mod}_\infty^{tf}(A_\hbar)), \quad \mathcal{H}_\infty^{tf}(B_\hbar) := \mathcal{H}(\mathbf{Mod}_\infty^{tf}(B_\hbar)).$$

The objects in $\mathcal{H}_\infty^{tf}(A_\hbar)$, resp. $\mathcal{H}_\infty^{tf}(B_\hbar)$ are the same objects of $\mathbf{Mod}_{tf}^\infty(A_\hbar)$, resp. $\mathbf{Mod}_{tf}^\infty(B_\hbar)$. The morphisms are equivalence classes w.r.t. the equivalence relation defined as follows; two morphisms $f_\hbar, g_\hbar : X_\hbar \rightarrow Y_\hbar$ in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$, resp. $\mathbf{Mod}_{tf}^\infty(B_\hbar)$ are equivalent, i.e. $f_\hbar \sim g_\hbar$, if there exists a strictly unital topological A_∞ -homotopy H_\hbar (see 8.3.1., subsubsection “On morphisms, quasi-isomorphisms and homotopy equivalences”) s.t. $f_\hbar - g_\hbar = d_{Y_\hbar} \circ H_\hbar + H_\hbar \circ d_{X_\hbar}$, denoting by d_{X_\hbar} , resp. d_{Y_\hbar} the codifferentials on X_\hbar , resp. Y_\hbar . \sim is an equivalence relation on morphisms in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$, resp. $\mathbf{Mod}_{tf}^\infty(B_\hbar)$. We want to prove that $\mathcal{H}_\infty^{tf}(A_\hbar)$ and $\mathcal{H}_\infty^{tf}(B_\hbar)$ are triangulated categories.

8.0.34. *Triangulated structure on $\mathcal{H}_\infty^{tf}(A_\hbar)$, $\mathcal{H}_\infty^{tf}(B_\hbar)$.* We endow the categories $\mathcal{H}_\infty^{tf}(A_\hbar)$ and $\mathcal{H}_\infty^{tf}(B_\hbar)$ with a triangulated structure such that, for $\hbar = 0$ it reduces to the triangulated structure on $\mathcal{H}_\infty(A)$ and $\mathcal{H}_\infty(B)$. We refer to Appendix A for the notation on triangulated categories. As usual we give the definition for $\mathcal{H}_\infty^{tf}(A_\hbar)$; it applies to $\mathcal{H}_\infty^{tf}(B_\hbar)$ as well, with due changes.

Let $0 \rightarrow M_\hbar \xrightarrow{f_\hbar} M'_\hbar \xrightarrow{g_\hbar} M''_\hbar \rightarrow 0$ be a short exact sequence of objects in $\mathcal{H}_\infty^{tf}(A_\hbar)$ with f_\hbar and g_\hbar strict. This means that, for any $(i, j) \in \mathbb{Z}^2$, then

$$0 \rightarrow (M_\hbar)_j^i \xrightarrow{f_\hbar} (M'_\hbar)_j^i \xrightarrow{g_\hbar} (M''_\hbar)_j^i \rightarrow 0$$

is short exact as sequence of $\mathbb{K}[[\hbar]]$ -modules.

Definition 29. *The triangulated structure on the additive category $\mathcal{H}_\infty^{tf}(A_\hbar)$ is given as follows. The endofunctor Σ is simply the (cohomological) grading shift functor $\Sigma = [1]$. The distinguished triangles are isomorphic to those induced by semi-split sequences of strict morphisms*

$$M_\hbar \xrightarrow{f_\hbar} M'_\hbar \xrightarrow{g_\hbar} M''_\hbar$$

in $\mathbf{Mod}_{tf}^\infty(A_\hbar)$, i.e. sequences such that $0 \rightarrow M_\hbar \xrightarrow{f_\hbar} M'_\hbar \xrightarrow{g_\hbar} M''_\hbar \rightarrow 0$ is an exact sequence in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$, and such that there exists a strict splitting

$$\rho_\hbar = \sum_{k \geq 0} \rho^{(k)} \hbar^k, \quad \rho^{(k)} = \rho^{(k),0} \in \mathrm{Hom}_{\mathbf{bG}_{\mathbb{K}}}^{0,0}(M'[1], M[1])$$

of f_\hbar , i.e.

$$(27) \quad \rho_\hbar \circ f_\hbar = 1_\hbar,$$

with

$$(28) \quad \rho_\hbar \circ d_{M_\hbar} = d_{M'_\hbar} \circ (\rho_\hbar \tilde{\otimes} 1_{\hbar}^{\otimes i-1}), \quad i \geq 2.$$

By the very definition if the triangulated structure on $\mathcal{H}_\infty^{tf}(A_\hbar)$ we have

Corollary 8. *The “evaluation at $\hbar = 0$ ” functor $(E_\hbar, 1)$, $E_\hbar : \mathcal{H}_\infty^{tf}(A_\hbar) \rightarrow \mathbf{Mod}_\infty(A)/\sim$ with $E_\hbar(M_\hbar) := M_\hbar/\hbar M_\hbar$, is exact w.r.t. the triangulated category structures on $\mathcal{H}_\infty^{tf}(A_\hbar)$ and $\mathbf{Mod}_\infty(A)/\sim$.*

8.0.35. *Characterization of exact triangles in $\mathcal{H}_\infty^{tf}(A_\hbar)$.* Before proving that the endofunctor [1] and the class of exact triangles in the above definition endow $\mathcal{H}_\infty^{tf}(A_\hbar)$ with a triangulated category structure, let us better characterize the exact triangles. What follows is a suitable topological A_∞ -version of the analysis contained in [20] on the triangulated structure of the homotopy category $\mathcal{K}(\mathcal{A})$ of any additive category \mathcal{A} . The goal is to show that it is possible to lift those computations to the aforementioned topological A_∞ -case.

8.0.36. *Cones and cylinders. Exact sequences of topologically free modules.* We recall that, given a topological A_∞ -module M_\hbar , the bigraded object $M_\hbar[\pm 1]$ can be endowed with a topological A_∞ -module structure *via*

$$\bar{d}_{M_\hbar[\pm 1]}^{(i),l} = -s \circ \bar{d}_{M_\hbar}^{(i),l} \circ (s^{-1} \otimes 1).$$

Let $f_\hbar : (M_\hbar, d_{M_\hbar}) \rightarrow (N_\hbar, d_{N_\hbar})$ be a morphism in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$; f_\hbar , d_{M_\hbar} and d_{N_\hbar} are uniquely determined by formal power series whose i -th components are $f_\hbar^{(i)}$, $d_{M_\hbar}^{(i)}$ and $d_{N_\hbar}^{(i)}$.

Definition 30. *A cone $C(f_\hbar)$ of f_\hbar is the object*

$$C(f_\hbar) := M_\hbar[1] \tilde{\oplus} N_\hbar$$

with topological A_∞ -structure given by the differential $d_{C(f_\hbar)}$, such that

$$(29) \quad D_{C(f_\hbar)} = \begin{pmatrix} d_{M_\hbar[1]} & 0 \\ s^{-1} \circ f_\hbar & s^{-1} \circ d_{N_\hbar} \circ s \end{pmatrix}$$

Definition 31. *The A_∞ -cylinder $\text{Cyl}(f_\hbar)$ is the object*

$$\text{Cyl}(f_\hbar) = M_\hbar \tilde{\oplus} M_\hbar[1] \tilde{\oplus} N_\hbar,$$

with codifferential $d_{\text{Cyl}(f_\hbar)}$ given by

$$(30) \quad D_{\text{Cyl}(f_\hbar)} = \begin{pmatrix} s^{-1} \circ d_{M_\hbar} \circ s & -i_\hbar \circ s^{-1} & 0 \\ 0 & d_{M_\hbar[1]} & 0 \\ 0 & s^{-1} \circ f_\hbar & s^{-1} \circ d_{N_\hbar} \circ s \end{pmatrix}.$$

The natural inclusion

$$i_\hbar : M_\hbar \rightarrow \text{Cyl}(f_\hbar), \quad i_\hbar = \sum_{i \geq 0} i_\hbar^{(i)} \hbar^i,$$

with

$$i_\hbar^{(i),n} = 0, \quad \text{for } i, n \geq 1,$$

and $i_\hbar^{(0),0} = i : M \rightarrow M \oplus M[1] \oplus N$, $m \mapsto (m, 0, 0)$, is a strict morphism of topological A_∞ - A_\hbar -modules.

Proposition 16. *For any morphism f_\hbar in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$*

$$d_{C(f_\hbar)} \circ d_{C(f_\hbar)} = d_{\text{Cyl}(f_\hbar)} \circ d_{\text{Cyl}(f_\hbar)} = 0.$$

Proof. Using (29) and (30), the proof is immediate. □

For any morphism $f_\hbar : (M_\hbar, d_{M_\hbar}) \rightarrow (N_\hbar, d_{N_\hbar})$ we consider the sequence

$$(31) \quad 0 \rightarrow M_\hbar \xrightarrow{i_\hbar} \text{Cyl}(f_\hbar) \xrightarrow{\pi_\hbar} C(f_\hbar) \rightarrow 0$$

The natural projection π_\hbar is the strict morphism of topological A_∞ -modules $\pi_\hbar = \sum_{i \geq 0} \pi_\hbar^{(i)} \hbar^i$, with $\pi_\hbar^{(i)} = 0$, for $i \geq 1$ and $\pi_\hbar^{(0),0} = \pi : M \oplus M[1] \oplus N \rightarrow M[1] \oplus N$. The sequence (31) is exact in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$; actually more can be said: as $\ker \pi_\hbar = M_\hbar = \text{im } i_\hbar$ then (31) is exact in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$.

Proposition 17. *Let (M_\hbar, d_{M_\hbar}) , (N_\hbar, d_{N_\hbar}) and (L_\hbar, d_{L_\hbar}) be objects in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$ and $f_\hbar : M_\hbar \rightarrow N_\hbar$, $g_\hbar : N_\hbar \rightarrow L_\hbar$ be strict morphisms in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$. Any short exact sequence*

$$0 \rightarrow M_\hbar \xrightarrow{f_\hbar} N_\hbar \xrightarrow{g_\hbar} L_\hbar \rightarrow 0$$

in $\mathbf{bG}_{\mathbb{K}[[\hbar]]}$ is quasi-isomorphic in $\mathbf{Mod}_\infty^{tf}(A_\hbar)$ to the short exact sequence $0 \rightarrow M_\hbar \xrightarrow{i_\hbar} \text{Cyl}(f_\hbar) \xrightarrow{\pi_\hbar} C(f_\hbar) \rightarrow 0$.

Proof. Like in [20], prop. 5, section III, with due changes. □

For any morphism $f_h : M_h \rightarrow N_h$ in $\mathbf{Mod}_\infty^{tf}(A_h)$ the sequence

$$0 \rightarrow M_h \xrightarrow{i_h} \mathrm{Cyl}(f_h) \xrightarrow{\pi_h} C(f_h) \rightarrow 0$$

is exact. Something more can be said; in fact the sequence is semi-split with strict splitting $\rho_h : \mathrm{Cyl}(f_h) \rightarrow M_h$ given by

$$\rho_h^{(0)}(m, sm', l) = m, \quad \rho_h^{(i)} = 0 \text{ for } i \geq 1.$$

It is important to note that ρ_h does not commute with the components $d_{\mathrm{Cyl}(f_h)}^{(i),0}$ and $d_{M_h}^{(i),0}$ of the codifferentials on $\mathrm{Cyl}(f_h)$ and M_h for any $i \geq 0$, but

$$\rho_h^{(0)}(s^{-1}(d_{\mathrm{Cyl}(f_h)}^{(j),n}(m, sm', l|a^{\otimes n}))) = d_{M_h}^{(j),n}(\rho_h^{(0)}(m, sm', l)|a^{\otimes n}), \text{ for any } n \geq 1.$$

Like in the classical case, the presence of the inclusion i_h in the definition of the codifferential $d_{\mathrm{Cyl}(f_h)}$ plays a major role. In summary,

$$M_h \xrightarrow{i_h} \mathrm{Cyl}(f_h) \xrightarrow{\pi_h} C(f_h) \rightarrow M_h[1]$$

is an exact triangle in $\mathcal{H}_\infty^{tf}(A_h)$ for any morphism $f_h : M_h \rightarrow L_h$ in $\mathbf{Mod}_\infty^{tf}(A_h)$.

Let

$$(32) \quad M_h \xrightarrow{f_h} L_h \xrightarrow{p'_h} C(f_h) \xrightarrow{r_h} M_h[1];$$

be a sequence in $\mathcal{H}_\infty^{tf}(A_h)$; it is isomorphic in $\mathcal{H}_\infty^{tf}(A_h)$ to the exact triangle $M_h \xrightarrow{i_h} \mathrm{Cyl}(f_h) \xrightarrow{\pi_h} C(f_h) \xrightarrow{r_h} M_h[1]$ via

$$(33) \quad \begin{array}{ccccccc} M_h & \xrightarrow{f_h} & L_h & \xrightarrow{p'_h} & C(f_h) & \xrightarrow{r_h} & M_h[1] \\ \downarrow 1_h & & \downarrow \alpha_h & & \downarrow 1_h & & \downarrow 1_h \\ M_h & \xrightarrow{i_h} & \mathrm{Cyl}(f_h) & \xrightarrow{\pi_h} & C(f_h) & \xrightarrow{r_h} & M_h[1] \end{array}$$

with strict A_∞ -morphism

$$\alpha_h : L_h \rightarrow \mathrm{Cyl}(f_h), \quad \alpha_h^{(0),0}(l) = (0, 0, l)$$

and $\alpha_h^{(i)} = 0$, for $i \geq 1$. In summary, (32) is an exact triangle in $\mathcal{H}_\infty^{tf}(A_h)$, as well.

8.0.37. *Other exact triangles: using the splitting.* Let

$$0 \rightarrow M_h \xrightarrow{f_h} N_h \xrightarrow{g'_h} Q_h \rightarrow 0$$

be a semi-split exact sequence like in def. 25, with splitting ρ_h and $N_h = N[[h]]$, $M_h = M[[h]]$, $Q_h = Q[[h]]$ in $\mathbf{bG}_{\mathbb{K}}[[h]]$.

At the order h^0 eq. (27) is equivalent to $\rho_h^{(0)} \circ f_h^{(0)} = 1$. This implies that $N_j^i \cong M_j^i \oplus Q_j^i$ as \mathbb{K} -modules, for any $(i, j) \in \mathbb{Z}^2$; in virtue of this we assume that $N_h = ((M \oplus Q)[[h]], d_{M_h \oplus Q_h})$, where

$$d_{M_h \oplus Q_h} = \begin{pmatrix} d_{M_h} & -f_h \\ 0 & d_{Q_h} \end{pmatrix}.$$

and $M_h \tilde{\oplus} Q_h \equiv (M \oplus Q)[[h]]$ in $\mathbf{bG}_{\mathbb{K}}[[h]]$. It follows that $d_{M_h \oplus Q_h} \circ d_{M_h \tilde{\oplus} Q_h} = 0$ if and only if

$$f_h : Q_h \rightarrow M_h[1]$$

defines an A_∞ -morphism of topological A_∞ - A_h -modules. By definition of the triangulated structure on $\mathcal{H}_\infty^{tf}(A_h)$, the sequence $M_h \xrightarrow{i_h} M_h \tilde{\oplus} Q_h \xrightarrow{p_h} Q_h \xrightarrow{f_h} M_h[1]$ is an exact triangle.

Theorem 8. *The homotopy categories $\mathcal{H}_\infty^{tf}(A_h)$ and $\mathcal{H}_\infty^{tf}(B_h)$ are triangulated; the triangulated structure is the one given in def. 25.*

Proof. [20], pag. 246, with due changes; we sketch the proof for sake of clarity. On the axiom (T1) (see the Appendix); the sequence

$$X_h \xrightarrow{1_h} X_h \rightarrow 0 \rightarrow X_h[1]$$

is isomorphic to $X_h \xrightarrow{1_h} X_h \rightarrow C(1_h) \rightarrow X_h[1]$ as the zero morphism $0 \rightarrow C(1_h)$ is homotopic to the identity morphism $1'_h : C(1_h) \rightarrow C(1_h)$ on $C(1_h)$; in fact

$$1'_h = H_h \circ d_{C(1_h)} + d_{C(1_h)} \circ H_h,$$

with strict topological A_∞ -homotopy $H_{\bar{h}} = H^{(0),0}$, $H^{(0),0}(sx \oplus x') = (x', 0)$. Compatibility with the codifferentials follows easily. Axiom (T2) is proved similarly. Let

$$X_{\bar{h}} \xrightarrow{u_{\bar{h}}} Y_{\bar{h}} \xrightarrow{v_{\bar{h}}} C(u_{\bar{h}}) \xrightarrow{p_{\bar{h}}} X_{\bar{h}}[1]$$

be an exact triangle. We want to prove that the sequence

$$Y_{\bar{h}} \xrightarrow{v_{\bar{h}}} C(u_{\bar{h}}) \xrightarrow{p_{\bar{h}}} X_{\bar{h}}[1] \xrightarrow{-u_{\bar{h}}[1]} Y_{\bar{h}}[1]$$

is isomorphic to the exact triangle

$$Y_{\bar{h}} \xrightarrow{v_{\bar{h}}} C(u_{\bar{h}}) \xrightarrow{s_{\bar{h}}} C(v_{\bar{h}}) \xrightarrow{-u_{\bar{h}}[1]} Y_{\bar{h}}[1].$$

All we need is to introduce the topological A_∞ -homotopy equivalence

$$\theta_{\bar{h}} : X_{\bar{h}}[1] \rightarrow C(v_{\bar{h}}),$$

with

$$\theta_{\bar{h}}^{(0),0}(sx) = (-su_{\bar{h}}^{(0),0}(sx), sx, 0), \quad \theta_{\bar{h}}^{(i),n}(x|a_1| \dots |a_n) = (-su_{\bar{h}}^{(i),n}((x|a_1| \dots |a_n)), 0, 0), \quad n \geq 1$$

and to check that $s_{\bar{h}} \circ 1_{\bar{h}} - \theta_{\bar{h}} \circ p_{\bar{h}} = d_{C(v_{\bar{h}})}H_{\bar{h}} + H_{\bar{h}}d_{C(u_{\bar{h}})}$ with strict A_∞ -homotopy $H_{\bar{h}} : C(u_{\bar{h}}) \rightarrow C(v_{\bar{h}})$, $H^{(0),0}(sx, y) = (y, 0, 0)$, $H^{(i),n} = 0$ otherwise.

The computations are long but straightforward; $\theta_{\bar{h}}$ is a homotopy equivalence because it admits the strict inverse

$$\psi_{\bar{h}} : C(v_{\bar{h}}) \rightarrow X_{\bar{h}}[1], \quad \psi_{\bar{h}}^{(0),0}(sy, sx, y') = sx,$$

and $\psi_{\bar{h}}^{(i),n} = 0$ otherwise. Clearly $\psi_{\bar{h}} \circ \theta_{\bar{h}} = 1_{\bar{h}}$, but $\theta_{\bar{h}} \circ \psi_{\bar{h}} = 1_{\bar{h}} + d_{C(v_{\bar{h}})}H'_{\bar{h}} + H'_{\bar{h}}d_{C(v_{\bar{h}})}$, with $H'^{(0),0}(sy, sx, y') = (y', 0, 0)$ and zero otherwise.

Axiom (T3) is proved by using cones and (T4) follows by using semi split exact sequences. \square

8.0.38. *Localizing w.r.t. topological A_∞ -quasi-isomorphisms: on the derived categories $\mathbf{D}_{tf}^\infty(A_{\bar{h}})$ and $\mathbf{D}_{tf}^\infty(B_{\bar{h}})$.* In [20], def.6, section III, localizing classes of morphisms are defined. In our setting we have

Proposition 18. *The class Qis of quasi-isomorphisms in the homotopic categories $\mathcal{H}_\infty^{tf}(A_{\bar{h}})$ and $\mathcal{H}_\infty^{tf}(B_{\bar{h}})$ is localizing.*

Proof. We prove the statement for $\mathcal{H}_\infty^{tf}(A_{\bar{h}})$. We refer to the proof of thm. 4, pag 160 in [20]. We “translate” it in our topological A_∞ -case, with due changes. \square

Thanks to the above proposition the following definition makes sense.

Definition 32. *The localizations*

$$\mathbf{D}_{tf}^\infty(A_{\bar{h}}) := \mathcal{H}_\infty^{tf}(A_{\bar{h}})[Qis^{-1}], \quad \text{resp.} \quad \mathbf{D}_{tf}^\infty(B_{\bar{h}}) := \mathcal{H}_\infty^{tf}(B_{\bar{h}})[Qis^{-1}]$$

are said to be the derived categories of $\mathbf{Mod}_\infty^{tf}(A_\infty)$, resp. $\mathbf{Mod}_\infty^{tf}(B_\infty)$.

The objects in $\mathbf{D}_{tf}^\infty(A_{\bar{h}})$, resp. $\mathbf{D}_{tf}^\infty(B_{\bar{h}})$ are the same objects of $\mathcal{H}_\infty^{tf}(A_{\bar{h}})$, resp. $\mathcal{H}_\infty^{tf}(B_{\bar{h}})$ while the morphisms are described through the equivalence classes of “roofs”, as in [20]. We use the notation $\mathcal{D} = \mathbf{D}_{tf}^\infty(A_{\bar{h}}), \mathbf{D}_{tf}^\infty(B_{\bar{h}})$. Any morphism $\phi_{\bar{h}} : X_{\bar{h}} \rightarrow Y_{\bar{h}}$ in \mathcal{D} is represented by an equivalence class of roofs; if two roofs $(s_{\bar{h}}, \bar{\phi}_{\bar{h}})$ and $(t_{\bar{h}}, \bar{\psi}_{\bar{h}})$ representing the same morphism in \mathcal{D} are equivalent, we will simply write $(s_{\bar{h}}, \bar{\phi}_{\bar{h}}) = (t_{\bar{h}}, \bar{\psi}_{\bar{h}})$.

In what follows we will state that the morphism $\phi_{\bar{h}} : X_{\bar{h}} \rightarrow Y_{\bar{h}}$ in \mathcal{D} is represented by *the* roof $(s_{\bar{h}}, \bar{\phi}_{\bar{h}})$, for simplicity. The identity morphism $1_{\bar{h}} : X_{\bar{h}} \rightarrow X_{\bar{h}}$ in \mathcal{D} is represented by

$$\begin{array}{ccc} & X_{\bar{h}} & \\ 1_{\bar{h}} \swarrow & & \searrow 1_{\bar{h}} \\ X_{\bar{h}} & & X_{\bar{h}} \end{array}$$

for any $X_{\bar{h}}$ in \mathcal{D} . The composition

$$(34) \quad \psi_{\bar{h}} \circ \phi_{\bar{h}}$$

of morphisms $\phi_{\bar{h}} : X_{\bar{h}} \rightarrow Y_{\bar{h}}$, $\psi_{\bar{h}} : Y_{\bar{h}} \rightarrow Z_{\bar{h}}$ in \mathcal{D} represented by the roofs $(s_{\bar{h}}, \bar{\phi}_{\bar{h}})$ and $(t_{\bar{h}}, \bar{\psi}_{\bar{h}})$ will be denoted also by

$$(t_{\bar{h}}, \bar{\psi}_{\bar{h}}) \circ (s_{\bar{h}}, \bar{\phi}_{\bar{h}}).$$

Corollary 9. *The class of quasi-isomorphisms in $\mathcal{H}_\infty^{tf}(A_h)$ and $\mathcal{H}_\infty^{tf}(B_h)$ is compatible with triangulation; the derived categories*

$$\mathbf{D}_{tf}^\infty(A_h), \mathbf{D}_{tf}^\infty(B_h).$$

are triangulated.

Proof. See [20]; the proofs there apply here with straightforward changes. \square

8.0.39. *On the quantized Functors.* Let us define the functors

$$F_h : \mathbf{Mod}_\infty^{tf}(A_h) \rightarrow \mathbf{Mod}_\infty^{tf,strict}(B_h), \quad G_h : \mathbf{Mod}_\infty^{tf}(B_h) \rightarrow \mathbf{Mod}_\infty^{tf,strict}(A_h),$$

via

$$F_h(M_h) := M_h \tilde{\otimes}_{A_h} K_h, \quad G_h(N_h) := N_h \tilde{\otimes}_{B_h} \underline{K}_h,$$

on objects $M_h \in \mathbf{Mod}_\infty^{tf}(A_h)$ and $N_h \in \mathbf{Mod}_\infty^{tf}(B_h)$. Let $f_h : M_h \rightarrow N_h$ be a morphism in $\mathbf{Mod}_\infty^{tf}(A)$. Then $F_h(f_h)$ is the strict morphism in $\mathbf{Mod}_\infty^{tf}(B)$ given by

$$F_h(f_h) = \sum_{i \geq 0} F_h^{(i)}(f_h) h^i, \quad F_h^{(i),0}(f_h) = \sum_{k \geq 0} f_h^{(i),k} \otimes 1.$$

and zero otherwise. Similar definition holds true for G'_h . Here $\mathbf{Mod}_\infty^{tf,strict}(A_h)$ denotes the subcategory of $\mathbf{Mod}_\infty^{tf}(A_h)$ with same objects and strict topological A_∞ -morphisms. Same convention holds true for $\mathbf{Mod}_\infty^{tf,strict}(B_h)$.

Lemma 13. *Let $f_h : M_h \rightarrow N_h$ be a quasi-isomorphism in $\mathbf{Mod}_\infty^{tf}(A_h)$; then $F_h(f_h) : M_h \tilde{\otimes}_{A_h} K_h \rightarrow N_h \tilde{\otimes}_{A_h} K_h$ is a quasi-isomorphism in $\mathbf{Mod}_\infty^{tf,strict}(A_h)$.*

Similar considerations hold for the functor G_h .

8.0.40. *The quantized functors on the derived categories; compatibility with the triangulated structures.* We discuss now the above quantized functors lifting them on the derived categories $\mathbf{D}_{tf}^\infty(A_h)$ and $\mathbf{D}_{tf}^\infty(B_h)$.

Definition 33. \mathcal{F}_h is the unique functor $\mathcal{F}_h : \mathbf{D}_{tf}^\infty(A_h) \rightarrow \mathbf{D}_{tf}^\infty(B_h)$ s.t.

$$\mathcal{F}_h \circ Q_{A_h} = \mathcal{T}_h,$$

denoting by $Q_{A_h} : \mathcal{H}_\infty^{tf}(A_h) \rightarrow \mathbf{D}_{tf}^\infty(A_h)$ the canonical functor

$$Q_{A_h}(X) = X, \quad Q_{A_h}(f_h) = (1, f_h),$$

and by $\mathcal{T}_h : \mathcal{H}_\infty^{tf}(A_h) \rightarrow \mathbf{D}_{tf}^\infty(B_h)$ the composition

$$\mathcal{T}_h = Q_{B_h} \circ \bar{F}_h,$$

where $\bar{F}_h : \mathcal{H}_\infty^{tf}(A_h) \rightarrow \mathcal{H}_\infty^{tf}(B_h)$ is the functor induced by F_h on the homotopy categories of $\mathbf{Mod}_\infty^{tf}(A_h)$ and $\mathbf{Mod}_\infty^{tf}(B_h)$.

The functor $\mathcal{G}_h : \mathbf{D}_{tf}^\infty(B_h) \rightarrow \mathbf{D}_{tf}^\infty(A_h)$ is defined similarly. By definition

$$\mathcal{F}_h(X_h) = \bar{F}_h(X_h) = F_h(X_h),$$

on every object $X_h \in \mathbf{D}_{tf}^\infty(A_h)$ and on any morphism (s_h, f_h) in $\mathbf{D}_{tf}^\infty(A_h)$:

$$(35) \quad \mathcal{F}_h(s_h, f_h) = (\bar{F}_h(s_h), \bar{F}_h(f_h)).$$

Both $(\mathcal{F}_h, \varphi_h^1)$ and $(\mathcal{G}_h, \varphi_h^2)$ are exact functors w.r.t. the triangulated structure on $\mathbf{D}_{tf}^\infty(A_h)$ and $\mathbf{D}_{tf}^\infty(B_h)$. Here $\varphi_h^1 : \mathcal{F}_h \circ [1] \rightarrow [1] \circ \mathcal{F}_h$ denotes the obvious morphism of functors, and similarly for φ_h^2 .

Proposition 19. *Let $(\mathcal{F}_h, \mathcal{G}_h)$ be the pair of functors defined above.*

- A_h is isomorphic to $\mathcal{G}_h(\mathcal{F}_h(A_h))$ in $\mathbf{D}_{tf}^\infty(A_h)$.
- K_h is isomorphic to $\mathcal{F}_h(\mathcal{G}_h(K_h))$ in $\mathbf{D}_{tf}^\infty(B_h)$.

Proof. The first isomorphism is represented by

$$\begin{array}{ccc} & A_h & \\ 1_h \swarrow & & \searrow \bar{\psi}_{A_h} \\ A_h & & \mathcal{G}_h(\mathcal{F}_h(A_h)) \end{array}$$

with

$$\bar{\psi}_{A_h} : A_h \rightarrow K_h \tilde{\otimes}_{B_h} K_h \rightarrow (A_h \tilde{\otimes}_{A_h} K_h) \tilde{\otimes}_{B_h} K_h = \mathcal{G}_h(\mathcal{F}_h(A_h));$$

the second isomorphism is represented by

$$\begin{array}{ccc} & K_h & \\ 1_h \swarrow & & \searrow \bar{\psi}_{K_h} \\ K_h & & \mathcal{F}_h(\mathcal{G}_h(K_h)) \end{array}$$

with

$$\bar{\psi}_{K_h} : K_h \rightarrow A_h \tilde{\otimes}_{A_h} K_h \rightarrow (K_h \tilde{\otimes}_{B_h} K_h) \tilde{\otimes}_{A_h} K_h = \mathcal{F}_h(\mathcal{G}_h(K_h))$$

$\bar{\psi}_{A_h}$ and $\bar{\psi}_{K_h}$ are defined in cor. 7. \square

9. MAIN RESULT

Denoting by $\mathbf{triang}_{A_h}^\infty(A_h)$ the triangulated subcategory of $\mathbf{D}_{tf}^\infty(A_h)$ generated by $A_h[i]\langle j \rangle$ and by $\mathbf{triang}_{B_h}^\infty(K_h)$ the triangulated subcategory of $\mathbf{D}_{tf}^\infty(B_h)$ generated by $K_h[i]\langle j \rangle$, $i, j \in \mathbb{Z}$, we arrive at the main result of these notes.

Theorem 9. *Let X be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$, or \mathbb{C} and (A, K, B) be the triple of bigraded A_∞ -structures introduced in section 6. By $\hbar\pi \in (T_{poly}(X)[[\hbar]], 0, [\cdot, \cdot]_h)$ we denote an \hbar -formal quadratic Poisson bivector on X such that (A_h, K_h, B_h) is the Deformation Quantization on (A, K, B) w.r.t. $\hbar\pi$. The functor*

$$\mathcal{F}_h : \mathbf{D}_{tf}^\infty(A_h) \rightarrow \mathbf{D}_{tf}^\infty(B_h), \quad \mathcal{F}_h(\bullet) = \bullet \tilde{\otimes}_{A_h} K_h$$

induces equivalences of triangulated categories

$$\mathbf{triang}_{A_h}^\infty(A_h) \simeq \mathbf{triang}_{B_h}^\infty(K_h), \quad \mathbf{thick}_{A_h}^\infty(A_h) \simeq \mathbf{thick}_{B_h}^\infty(K_h).$$

Let $(\tilde{K}, d_{\tilde{K}})$ be the A_∞ - B - A -bimodule with $\tilde{K} = K$ and $d_{\tilde{K}}$ obtained from d_K exchanging A and B and $(\tilde{K}_h, d_{\tilde{K}_h})$ be its quantization w.r.t. π_h ; the functor

$$\mathcal{F}_h'' : \mathbf{D}_{tf}^\infty(B_h) \rightarrow \mathbf{D}_{tf}^\infty(A_h), \quad \mathcal{F}_h''(\bullet) = \bullet \tilde{\otimes}_{B_h} \tilde{K}_h$$

induces the equivalence of triangulated categories

$$\mathbf{triang}_{A_h}^\infty(\tilde{K}_h) \simeq \mathbf{triang}_{B_h}^\infty(B_h), \quad \mathbf{thick}_{A_h}^\infty(\tilde{K}_h) \simeq \mathbf{thick}_{B_h}^\infty(B_h).$$

APPENDIX A. PROOF OF PROP. 6

- On the quadratic relations $d_{\underline{\text{End}}_B(K)}^2 = 0$.

First of all we note that the maps $D_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi)$ and $D_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | a_1 | \dots | a_m)$ have cohomological degree 2; we have already remarked that $\mathcal{L}_A(a_1 | \dots | a_n)$ has cohomological degree 1, instead. The relations $\bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(s\varphi)) = 0$ are immediate to prove. We prove the case $n \geq 2$, $m = 0$, i.e.

$$\begin{aligned} & \sum_{j=1}^n (-1)^{\sum_{i=1}^{j-1} (|a_i| - 1)} \bar{d}_{\underline{\text{End}}_B(K)}^{n-1,0}(a_1 | \dots | a_{j-1}, \bar{d}_A^2(a_j | a_{j+1}) | a_{j+2} | \dots | a_n | \varphi) + \\ & \sum_{n'=1}^{n-1} (-1)^{\sum_{i=1}^{n-n'} (|a_i| - 1)} \bar{d}_{\underline{\text{End}}_B(K)}^{n-n',0}(a_1 | \dots | a_{n-n'}, \bar{d}_{\underline{\text{End}}_B(K)}^{n',0}(a_{n-n'+1} | \dots | a_n | \varphi) + \\ & \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\bar{d}_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi)) + (-1)^{\sum_{i=1}^n (|a_i| - 1)} \bar{d}_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n, \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\varphi)) = 0; \end{aligned}$$

such quadratic relations are equivalent to

$$\begin{aligned} & \sum_{j=1}^n (-1)^{\sum_{i=1}^{j-1} (|a_i| - 1) + \sum_{i=1}^n (|a_i| - 1)} \mathcal{L}_A(a_1 | \dots | a_{j-1}, \bar{d}_A^2(a_j | a_{j+1}) | a_{j+2} | \dots | a_n) \circ \varphi + \\ & \sum_{n'=1}^{n-1} (-1)^{\sum_{i=1}^{n-n'} (|a_i| - 1)} \mathcal{L}_A(a_1 | \dots | a_{n-n'}) \circ (\mathcal{L}_A(a_{n-n'+1} | \dots | a_n) \circ \varphi) + \\ & (-1)^{\sum_{i=1}^n (|a_i| - 1)} \partial_{\underline{\text{End}}_B(K)}(\mathcal{L}_A(a_1 | \dots | a_n) \circ \varphi) + \mathcal{L}_A(a_1 | \dots | a_n) \circ \partial_{\underline{\text{End}}_B(K)}(\varphi) = 0 \end{aligned}$$

(36)

The last contributions on the l.h.s. of (36) can be written as

$$\begin{aligned} & (-1)^{\sum_{i=1}^n (|a_i|-1)} \partial_{\underline{\text{End}}_B(K)}(\mathcal{L}_A(a_1 | \dots | a_n) \circ \varphi) + \mathcal{L}_A(a_1 | \dots | a_n) \circ \partial_{\underline{\text{End}}_B(K)}(\varphi) = \\ & (-1)^{\sum_{i=1}^n (|a_i|-1)} \partial_{\underline{\text{End}}_B(K)}(\mathcal{L}_A(a_1 | \dots | a_n)) \circ \varphi = \\ & (\mathcal{L}_A(a_1 | \dots | a_n) \circ d_K + (-1)^{\sum_{i=1}^n (|a_i|-1)} d_K \circ \mathcal{L}_A(a_1 | \dots | a_n)) \circ \varphi, \end{aligned}$$

because $|\mathcal{L}_A(a_1 | \dots | a_n)| = \sum_{i=1}^n (|a_i| - 1) + 1$.

At the end, multiplying both sides of (36) by $(-1)^{\sum_{i=1}^n (|a_i|-1)}$ and using the associativity of the product \circ we get that (36) is equivalent to a finite sum of equations of the type $d_K^2(a_1 | \dots | a_n | \varphi(\dots)) = 0$.

We continue with the case $n = 0$, $m \geq 2$, i.e.

$$\begin{aligned} & \sum_{j=1}^m (-1)^{|\varphi|-1+\sum_{i=1}^{j-1} (|a_i|-1)} \bar{d}_{\underline{\text{End}}_B(K)}^{0,m-1}(\varphi | a_1 | \dots | a_{j-1}, \bar{d}_A^2(a_j | a_{j+1}) | a_{j+2} | \dots | a_m) + \\ & \sum_{m'=1}^{m-1} \bar{d}_{\underline{\text{End}}_B(K)}^{0,m-m'}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,m'}(\varphi | a_1 | \dots | a_{m'}), a_{m'+1} | \dots | a_m) + \\ & \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | a_1 | \dots | a_m)) + \bar{d}_{\underline{\text{End}}_B(K)}^{0,m}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\varphi), a_1 | \dots | a_m) = 0. \end{aligned}$$

The above relations are equivalent to

$$\begin{aligned} & \sum_{j=1}^m (-1)^{\sum_{i=1}^{j-1} (|a_i|-1)} \varphi \circ \mathcal{L}_A(a_1 | \dots | a_{j-1}, \bar{d}_A^2(a_j | a_{j+1}) | a_{j+2} | \dots | a_m) + \\ & \sum_{m'=1}^{m-1} (-1)^{\sum_{i=1}^{m-m'} (|a_i|-1)} \varphi \circ (\mathcal{L}_A(a_1 | \dots | a_{m-m'}) \circ \mathcal{L}_A(a_{m-m'+1} | \dots | a_m)) + \\ & (-1)^{|\varphi|+1} \partial_{\underline{\text{End}}_B(K)}(\varphi \circ \mathcal{L}_A(a_1 | \dots | a_m)) + (-1)^{|\varphi|} \partial_{\underline{\text{End}}_B(K)}(\varphi) \circ \mathcal{L}_A(a_1 | \dots | a_m) = 0, \end{aligned}$$

which are easily verified, as

$$\begin{aligned} & (-1)^{|\varphi|+1} \partial_{\underline{\text{End}}_B(K)}(\varphi \circ \mathcal{L}_A(a_1 | \dots | a_m)) + (-1)^{|\varphi|} \partial_{\underline{\text{End}}_B(K)}(\varphi) \circ \mathcal{L}_A(a_1 | \dots | a_m) = \\ & -\varphi \circ \partial_{\underline{\text{End}}_B(K)}(\mathcal{L}_A(a_1 | \dots | a_m)) = \\ & \varphi \circ \left((-1)^{\sum_{i=1}^n (|a_i|-1)+1} \mathcal{L}_A(a_1 | \dots | a_m) \circ d_K - d_K \circ \mathcal{L}_A(a_1 | \dots | a_m) \right). \end{aligned}$$

Note the overall -1 sign, which plays no role.

The equations expressing compatibility between the left and right actions on $\underline{\text{End}}_B(K)$ (for $n, m \geq 1$) are

$$\begin{aligned} & (-1)^{\sum_{i=1}^n (|a_i|-1)} \bar{d}_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n, \bar{d}_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | \bar{a}_1 | \dots | \bar{a}_m)) + \\ & \bar{d}_{\underline{\text{End}}_B(K)}^{0,m}(\bar{d}_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n, |\varphi| \bar{a}_1 | \dots | \bar{a}_m)) = 0; \end{aligned}$$

they are equivalent to

$$\begin{aligned} & \mathcal{L}_A(a_1 | \dots | a_n) \circ D_{\underline{\text{End}}_B(K)}^{0,m}(\varphi | \bar{a}_1 | \dots | \bar{a}_m) + \\ & (-1)^{|\varphi|+\sum_{i=1}^n (|a_i|-1)} D_{\underline{\text{End}}_B(K)}^{n,0}(a_1 | \dots | a_n | \varphi) \circ \mathcal{L}_A(\bar{a}_1 | \dots | \bar{a}_m) = 0, \end{aligned}$$

or

$$\begin{aligned} & (-1)^{|\varphi|+1} \mathcal{L}_A(a_1 | \dots | a_n) \circ (\varphi \circ \mathcal{L}_A(\bar{a}_1 | \dots | \bar{a}_m)) + \\ & (-1)^{|\varphi|} (\mathcal{L}_A(a_1 | \dots | a_n) \circ \varphi) \circ \mathcal{L}_A(\bar{a}_1 | \dots | \bar{a}_m) = 0. \end{aligned}$$

We finish by checking the compatibility of the actions with the differential, i.e.

$$(37) \quad (-1)^{|a|-1} \bar{d}_{\underline{\text{End}}_B(K)}^{1,0}(sa, \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\varphi)) + \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\bar{d}_{\underline{\text{End}}_B(K)}^{1,0}(a | \varphi)) = 0,$$

and

$$(38) \quad \bar{d}_{\underline{\text{End}}_B(K)}^{0,1}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\varphi) | a) + \bar{d}_{\underline{\text{End}}_B(K)}^{0,0}(\bar{d}_{\underline{\text{End}}_B(K)}^{0,1}(\varphi | a)) = 0;$$

(37) is equivalent to

$$\mathcal{L}_A(sa) \circ \partial_{\underline{\text{End}}_B(K)}(\varphi) + (-1)^{|a|-1} \partial_{\underline{\text{End}}_B(K)}(\mathcal{L}_A(sa) \circ \varphi) = 0;$$

(38) gives

$$\partial_{\underline{\text{End}}_B(K)}(\varphi \circ \mathcal{L}_A(sa)) - \partial_{\underline{\text{End}}_B(K)}(\varphi) \circ \mathcal{L}_A(sa) = 0.$$

Both relations are satisfied by checking that

$$\partial_{\text{End}_B(K)}(\mathcal{L}_A(sa)) = 0.$$

- L_A descends to a morphism of A_∞ - A - A -bimodules.

We prove that

$$(39) \quad L_A \circ \tilde{d}_A = d_{\text{End}_B(K)} \circ L_A.$$

We check (39) on strings $(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l) \in A[1]^{\otimes k+l+1}$ and $(1 | b_1 | \dots | b_q) \in K[1] \otimes B[1]^{\otimes q}$, for any $k, l, q \geq 0$. If $k, l \geq 1$, the l.h.s of (39) is

$$\begin{aligned} & \sum_{j=1}^k (-1)^{\sum_{i=1}^{j-1} (|a_i|-1)} d_K^{k+l,q}(a_1 | \dots | a_{j-1}, d_A(a_j | a_{j+1}) | a_{j+2} | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_q) + \\ & d_K^{k+l,q}(-1)^{\sum_{i=1}^{k-1} (|a_i|-1)}(a_1 | \dots | a_{k-1}, d_A(a_k | \bar{a}) | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_q) + \\ & d_K^{k+l,q}(-1)^{\sum_{i=1}^k (|a_i|-1)}(a_1 | \dots | a_k, d_A(\bar{a} | \tilde{a}_1) | \tilde{a}_2 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_q) + \\ & \sum_{j=1}^k (-1)^{\sum_{i=1}^k (|a_i|-1) + |\bar{a}|-1 + \sum_{i=1}^{j-1} (|\tilde{a}_i|-1)} \\ & d_K^{k+l,q}(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_{j-1}, d_A(\tilde{a}_j | \tilde{a}_{j+1}) | \tilde{a}_{j+2} | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_q). \end{aligned}$$

As $L_A(a_1 | \dots | a_n) = s \circ \mathcal{L}_A(a_1 | \dots | a_n)$ then the r.h.s. of (39) is given by the terms

$$\begin{aligned} & \bar{d}_{\text{End}_B(K)}^{0,0}(L_A(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l))(1 | b_1 | \dots | b_q) = (-1)^{\sum_{i=1}^k (|a_i|-1) + |\bar{a}|-1 + \sum_{i=1}^l (|\tilde{a}_i|-1) + 1} \\ & \left(d_K^{k+l+1,q-1}(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l, d_K^{0,1}(1 | b_1) | b_2 | \dots | b_q) + \right. \\ & \left. \sum_{j=1}^q (-1)^{1 + \sum_{i=1}^{j-1} (|b_i|-1)} d_K^{k+l+1,q-1}(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_{j-1}, d_B^2(b_j | b_{j+1}) | b_2 | \dots | b_q) \right) + \\ & -d_K^{0,1}(d_K^{k+l+1,q-1}(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | b_2 | \dots | b_{q-1}) | b_q), \end{aligned}$$

the sum over $k' \in \{1, \dots, k\}$ of terms

$$\begin{aligned} & \bar{d}_{\text{End}_B(K)}^{k',0}(a_1 | \dots | a_{k'}, L_A(a_{k'+1} | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l))(1 | b_1 | \dots | b_q) = \\ & \sum_{q'=1}^q (-1)^{\sum_{i=1}^{k'} (|a_i|-1) + 1} d_K^{k',q-q'}(a_1 | \dots | a_{k'}, \\ & d_K^{k-k'+l+1,q'}(a_{k'+1} | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_{q'}) | b_{q'+1} | \dots | b_q), \end{aligned}$$

and the sum over $l' \in \{0, \dots, l-1\}$ of terms

$$\begin{aligned} & \bar{d}_{\text{End}_B(K)}^{0,l-l'}(L_A(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_{l'} | \tilde{a}_{l'+1} | \dots | \tilde{a}_l))(1 | b_1 | \dots | b_q) = \\ & \sum_{q'=0}^q (-1)^{\sum_{i=1}^k (|a_i|-1) + |\bar{a}|-1 + \sum_{i=1}^{l'} (|\tilde{a}_i|-1) + 1} d_K^{k+l'+1,q-q'}(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_{l'}, \\ & d_K^{l-l'+q'}(\tilde{a}_{l'+1} | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_{q'}) | b_{q'+1} | \dots | b_q), \end{aligned}$$

i.e. those contributions in the r.h.s. of (39) corresponding to the right actions on elements in $\text{End}_B(K)$.

Moving the terms on the r.h.s. of (39) (note the overall -1 sign) to the l.h.s, we get that (39) are equivalent to

$$d_K^2(a_1 | \dots | a_k | \bar{a} | \tilde{a}_1 | \dots | \tilde{a}_l | 1 | b_1 | \dots | b_q) = 0.$$

The other cases, i.e. $k=0, l \geq 1, k \geq 1, l=0$ and $k=l=0$ are a trivial sign check. We are done.

APPENDIX B. PROOF OF THM. 7

The proof of thm. 7 is shown in detail. We note that all the proof is based on checking the commutativity of diagrams in which objects belonging the classes $\mathcal{S}_1, \mathcal{S}'_1$ appear (see below). Commutativity of the other diagrams follows from these two special case. Moreover, we do not need to perform any explicit computation; we just need to apply the definition of the A_∞ -morphisms we introduced in section 6.

Proof. On objects in $\mathcal{S}_1, \mathcal{S}'_1$. We introduce $\mathcal{S}_1 = \{A\langle i \rangle[n], i, n \in \mathbb{Z}\}$, and $\mathcal{S}'_1 = \{K\langle i \rangle[n], i, n \in \mathbb{Z}\}$. By definition of the functors $(\mathcal{F}, \mathcal{G})$ and by proposition 12 and 9 we get $\mathcal{G}(\mathcal{F}(X)) \simeq X$, $\mathcal{F}(\mathcal{G}(Y)) \simeq Y$, for every $X \in \mathcal{S}_1, Y \in \mathcal{S}'_1$, with $\mathcal{F}(X) \in \mathcal{S}'_1$ for every $X \in \mathcal{S}_1$, and $\mathcal{G}(Y) \in \mathcal{S}_1$ for every $Y \in \mathcal{S}'_1$.

On morphisms of objects in \mathcal{S}_1 .

We want to prove that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \varphi_X & & \downarrow \varphi_Y \\ \mathcal{G}(\mathcal{F}(X)) & \xrightarrow{\mathcal{G}(\mathcal{F}(f))} & \mathcal{G}(\mathcal{F}(Y)) \end{array}$$

commutes, for every X and Y in \mathcal{S}_1 and natural quasi-isomorphisms φ_X, φ_Y in $\mathbf{D}^\infty(A)$.

Let $f : A\langle k_1 \rangle[i] \rightarrow A\langle k_2 \rangle[j]$ be a morphism in $\mathbf{D}^\infty(A)$, for any $k_1, k_2, i, j \in \mathbb{Z}$. As usual $\bar{f}_n : (A\langle k_1 \rangle[i])[1] \otimes A[1]^{\otimes n} \rightarrow (A\langle k_2 \rangle[j])[1]$ denotes its n -th Taylor component, for $n \geq 0$. As $A[1]$ is concentrated in cohomological degree -1 and the morphism f is of bidegree $(0, 0)$, then $\bar{f}_n \neq 0$ if and only if $n = j - i$, i.e. there exists one and only one non trivial Taylor component, if $j - i \geq 0$. We denote by $\bar{f}_n = s \circ f_n \circ s^{-1}$ its desuspension. If $n = 0$, then f_0 is a right A -linear map.

If $X = A\langle k_1 \rangle[i]$ and $Y = A\langle k_2 \rangle[j]$ we need to check the commutativity of the diagram

$$(40) \quad \begin{array}{ccc} A\langle k_1 \rangle[i] & \xrightarrow{f_{j-i}} & A\langle k_2 \rangle[j] \\ \downarrow \mathcal{V}_1 & & \downarrow \mathcal{V}_2 \\ (A \otimes_A A)\langle k_1 \rangle[i] & & (A \otimes_A A)\langle k_2 \rangle[j] \\ \downarrow \mathcal{V}_3 & & \downarrow \mathcal{V}_4 \\ (A \otimes_A \underline{\text{End}}_B K)\langle k_1 \rangle[i] & & (A \otimes_A \underline{\text{End}}_B K)\langle k_2 \rangle[j] \\ \downarrow \mathcal{V}_5 & & \downarrow \mathcal{V}_6 \\ (A \otimes_A (K \otimes_B \underline{K}))\langle k_1 \rangle[i] & & (A \otimes_A (K \otimes_B \underline{K}))\langle k_2 \rangle[j] \\ \downarrow \parallel & & \downarrow \parallel \\ A\langle k_1 \rangle[i] \otimes_A (K \otimes_B \underline{K}) & \xrightarrow{\mathcal{G}(\mathcal{F}(f))} & A\langle k_2 \rangle[j] \otimes_A (K \otimes_B \underline{K}) \end{array}$$

Let us describe it in some detail. We give the definitions of the maps \mathcal{V}_i , up to suspensions and desuspensions w.r.t. the cohomological and internal degree. The strict quasi-isomorphism \mathcal{V}_1 is induced by the A_∞ quasi-isomorphism $\Phi : A \rightarrow A \otimes_A A$, described in lem. 3. A similar formula holds true for \mathcal{V}_2 . The quasi-isomorphism \mathcal{V}_3 is given by $\mathcal{V}_3 = 1 \otimes L_A$ and the morphism \mathcal{V}_4 is defined similarly. \mathcal{V}_5 (and similarly \mathcal{V}_6) is

$$\mathcal{V}_5 = 1 \otimes \mathcal{T}$$

where \mathcal{T} is the homotopy equivalence of A_∞ - A - A -bimodules $\mathcal{T} : \underline{\text{End}}_B(K) \rightarrow \mathbf{End}_B(\mathcal{B}_B(K)) \rightarrow K \otimes_B \underline{K}$ given in prop. 7 and prop. 8.

Let us prove commutativity of (40). The morphism f has a unique non trivial Taylor component f_n , for $n = j - i \geq 0$. For any

$$(a, a_1, \dots, a_{n'}) \in A \otimes A^{\otimes n'},$$

we distinguish the following cases.

- $n' < n$. Going east and then south in (40) we get 0; going south-east, instead, we arrive at

$$\mathcal{G}(\mathcal{F}(f))(a, a_1, \dots, a_{n'}, \mathcal{T}(L_A(1))) = 0,$$

because L_A is strictly unital (here we write $L_A(1) = L_A^1(1)$); all Taylor components $L_A^{m'+m''+1}(\dots|1|\dots)$ such that with $1 \leq m' + m''$ identically vanish.

- $n' = n$. Going east and then south in (40) we arrive at

$$(41) \quad f_n(a, a_1 | \dots | a_n) \otimes \mathcal{T}(L_A(1)),$$

as

$$\mathcal{V}_2(f_n(a, a_1, \dots, a_n)) = f_n(a, a_1, \dots, a_n) \otimes 1 \in (A \otimes A)_r^0 \subset (A \otimes_A A)_r^0,$$

denoting by $r \geq 0$ the internal degree of the string (a, a_1, \dots, a_n) . Going south-east in (40) we arrive at (41) as well. In fact

$$\mathcal{V}_1(a, a_1, \dots, a_n) = (a, a_1 | \dots | a_n, 1) + \sum_{n'=0}^{n-1} (\mathcal{V}_1^{n'}(a, a_1, \dots, a_{n'}) \otimes a_{n'+1}, \dots, a_n) \in A \underline{\otimes}_A A,$$

and

$$\mathcal{G}(\mathcal{F}(f))(\mathcal{V}_5(\mathcal{V}_3(\mathcal{V}_1(a, a_1 | \dots | a_n, 1))) = \mathcal{G}(\mathcal{F}(f))((a, a_1 | \dots | a_n) \otimes \mathcal{T}(\mathcal{L}_A(1)));$$

this is because \mathcal{L}_A is strictly unital and $\mathcal{G}(\mathcal{F}(f))$ strict. The diagram (40) commutes.

- $n' > n$. Going east and then south in (40) we arrive at

$$\mathcal{G}(\mathcal{F}(f))(a, a_1, \dots, a_n, a_{n+1}, \dots, a_{n'}, \mathcal{T}(\mathcal{L}_A(1))) = f_n(a, a_1, \dots, a_n), a_{n+1}, \dots, a_{n'}, \mathcal{T}(\mathcal{L}_A(1)),$$

as $\mathcal{G}(\mathcal{F}(f))$ is strictly unital.

In summary, (40) commutes.

Induction: \mathcal{S}_r

We denote by

$$\mathcal{S}_r = \underbrace{\mathcal{S}_1 \star \dots \star \mathcal{S}_1}_{r\text{-times}},$$

the r^{th} extensions of \mathcal{S}_1 , for every $r \geq 1$. We want to prove the commutativity of any diagram

$$(42) \quad \begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ \downarrow \varphi_X & & \downarrow \varphi_Y \\ \mathcal{G}(\mathcal{F}(X)) & \xrightarrow{\mathcal{G}(\mathcal{F}(f_1))} & \mathcal{G}(\mathcal{F}(Y)) \end{array}$$

with $X, Y \in \mathbf{triang}_A^\infty(A)$ and φ_X, φ_Y isomorphisms in $\mathbf{D}^\infty(A)$. If $X, Y \in \mathbf{triang}_A^\infty(A)$, then by definition there exist $r, r' \geq 1$ s.t. $X \in \mathcal{S}_r, Y \in \mathcal{S}_{r'}$ and exact triangles

$$X_1 \rightarrow X \rightarrow X'_{r-1} \xrightarrow{f} X_1[1]$$

and

$$Y_1 \rightarrow Y \rightarrow Y'_{r'-1} \xrightarrow{g} Y_1[1]$$

in $\mathbf{D}^\infty(A)$ for some morphisms f and g and $X_1, Y_1 \in \mathcal{S}_1, X'_{r-1} \in \mathcal{S}_{r-1}$ and $Y'_{r'-1} \in \mathcal{S}_{r'-1}$. They are isomorphic to the exact triangles $X_1 \xrightarrow{i_{X_1}} X_1 \oplus X'_{r-1} \xrightarrow{p_{X'_{r-1}}} X'_{r-1} \xrightarrow{f} X_1[1]$ and $Y_1 \xrightarrow{i_{Y_1}} Y_1 \oplus Y'_{r'-1} \xrightarrow{p_{Y'_{r'-1}}} Y'_{r'-1} \xrightarrow{g} Y_1[1]$ for some isomorphisms $\rho_X : X \rightarrow X_1 \oplus X'_{r-1}$ in $\mathbf{triang}_A^\infty(A)$ and $\rho_Y : Y \rightarrow Y_1 \oplus Y'_{r'-1}$ in $\mathbf{triang}_A^\infty(A)$. Commutativity of (42) is equivalent to the commutativity of

$$(43) \quad \begin{array}{ccc} X_1 \oplus X'_{r-1} & \xrightarrow{\tilde{f}_1} & Y_1 \oplus Y'_{r'-1} \\ \downarrow \varphi_{X_1 \oplus X'_{r-1}} & & \downarrow \varphi_{Y_1 \oplus Y'_{r'-1}} \\ \mathcal{G}(\mathcal{F}(X_1 \oplus X'_{r-1})) & \xrightarrow{\mathcal{G}(\mathcal{F}(\tilde{f}_1))} & \mathcal{G}(\mathcal{F}(Y_1 \oplus Y'_{r'-1})) \end{array}$$

where

$$\tilde{f}_1 = \rho_Y \circ f_1 \circ \rho_X^{-1}, \quad \varphi_X = \mathcal{G}(\mathcal{F}(\rho_X))^{-1} \circ \varphi_{X \oplus X'_{r-1}} \circ \rho_X$$

and similarly for φ_Y . Let us discuss (43) and the isomorphisms

$$\varphi_{X_1 \oplus X'_{r-1}} : X_1 \oplus X'_{r-1} \rightarrow \mathcal{G}(\mathcal{F}(X_1 \oplus X'_{r-1})) \equiv \mathcal{G}(\mathcal{F}(X_1)) \oplus \mathcal{G}(\mathcal{F}(X'_{r-1})),$$

$$\varphi_{Y_1 \oplus Y'_{r'-1}} : Y_1 \oplus Y'_{r'-1} \rightarrow \mathcal{G}(\mathcal{F}(Y_1 \oplus Y'_{r'-1})) \equiv \mathcal{G}(\mathcal{F}(Y_1)) \oplus \mathcal{G}(\mathcal{F}(Y'_{r'-1}));$$

we distinguish two cases.

- If $r = 2$, and $r' = 2$, then such isomorphisms are simply

$$\begin{aligned} \varphi_{X_1 \oplus X'_1} &:= \varphi_{X_1} \oplus \varphi_{X'_1}, \\ \varphi_{Y_1 \oplus Y'_1} &:= \varphi_{Y_1} \oplus \varphi_{Y'_1}; \end{aligned}$$

we have $\varphi_{X_1 \oplus X'_1} \circ d_{X_1 \oplus X'_1} = d_{\mathcal{G}(\mathcal{F}(X_1 \oplus X'_1))} \circ \varphi_{X_1 \oplus X'_1}$, i.e.

$$(44) \quad \varphi_{X_1 \oplus X'_1} \circ \begin{pmatrix} d_{X_1} & -f \\ 0 & d_{X'_1} \end{pmatrix} = \begin{pmatrix} d_{\mathcal{G}(\mathcal{F}(X_1))} & -\mathcal{G}(\mathcal{F}(f)) \\ 0 & d_{\mathcal{G}(\mathcal{F}(X'_1))} \end{pmatrix} \circ \varphi_{X_1 \oplus X'_1}$$

as φ_{X_1} and $\varphi_{X'_1}$ are morphisms in $\mathbf{triang}_A^\infty(A)$ such that $\mathcal{G}(\mathcal{F}(f)) \circ \varphi_{X'_1} = \varphi_{X_1} \circ f$ holds true: we proved this last equality in the previous subsection, for any $X_1, X'_1 \in \mathcal{S}_1$ and morphism $f : X'_1 \rightarrow X_1[1]$. Similar considerations hold for $\varphi_{Y_1 \oplus Y'_1}$. Moreover $\varphi_{X_1 \oplus X'_1}$ and $\varphi_{Y_1 \oplus Y'_1}$ are homotopy equivalences as φ_{X_1} , $\varphi_{X'_1}$, φ_{Y_1} and $\varphi_{Y'_1}$ are homotopy equivalences. We recall that

$$\varphi_X : X \rightarrow \mathcal{G}(\mathcal{F}(X))$$

is explicitly given (up to suspensions and desuspensions) by

$$(45) \quad \varphi_X(x|a_1| \dots |a_n) = \sum_{n'=0}^n ((x, a_1| \dots |a_{n'}|\mathcal{T}(\mathcal{L}_A(1)))|a_{n'+1}| \dots |a_n),$$

for any $X \in \mathcal{S}_1$. We are left to prove

$$(46) \quad \varphi_{Y_1 \oplus Y'_1} \circ \tilde{f}_1 = \mathcal{G}(\mathcal{F}(\tilde{f}_1)) \circ \varphi_{X_1 \oplus X'_1}.$$

The strategy is clear: we use the same techniques introduced in the previous subsection, for the \mathcal{S}_1 case. We just need to consider any string

$$((x_1 \oplus x'_1)|a_1| \dots |a_n) \in (X_1 \oplus X'_1)[1] \otimes A[1]^{\otimes n};$$

(46) is equivalent to

$$(47) \quad \begin{aligned} & \sum_{n' \geq 0} \varphi_{Y_1 \oplus Y'_1}(\tilde{f}_{1,n'}((x_1 \oplus x'_1)|a_1| \dots |a_{n'}|a_{n'+1}| \dots |a_n)) = \\ & \sum_{n' \geq 0} \left[\varphi_{Y_1}(\tilde{f}_{1,n'}^{Y_1}((x_1 \oplus x'_1)|a_1| \dots |a_{n'}|a_{n'+1}| \dots |a_n)) \oplus \right. \\ & \quad \left. \varphi_{Y'_1}(\tilde{f}_{1,n'}^{Y'_1}((x_1 \oplus x'_1)|a_1| \dots |a_{n'}|a_{n'+1}| \dots |a_n)) \right] \stackrel{!}{=} \\ & \mathcal{G}(\mathcal{F}(\tilde{f}_1))(\varphi_{X_1}(x_1|a_1| \dots |a_n)) \oplus \mathcal{G}(\mathcal{F}(\tilde{f}_1))(\varphi_{X'_1}(x'_1|a_1| \dots |a_n)) = \\ & \mathcal{G}(\mathcal{F}(\tilde{f}_1))(x_1|a_1| \dots |a_n, \mathcal{T}(\mathcal{L}_A(1))) \oplus \mathcal{G}(\mathcal{F}(\tilde{f}_1))(x'_1|a_1| \dots |a_n, \mathcal{T}(\mathcal{L}_A(1))), \end{aligned}$$

where $\tilde{f}_{1,n'}^{Y_1}$, resp. $\tilde{f}_{1,n'}^{Y'_1}$, denotes the projection of $\tilde{f}_{1,n'}$ onto Y_1 , resp. Y'_1 . In the last equality in (47) we have used the definition of φ_{X_1} , $\varphi_{X'_1}$, following (45); note that $\mathcal{G}(\mathcal{F}(\tilde{f}_1))$ is a strict morphism; in fact $\mathcal{G}(\mathcal{F}(\tilde{f}_1)) = \tilde{f}_1 \otimes 1$.

By definition (45), φ_{Y_1} and $\varphi_{Y'_1}$ leave the contributions $\tilde{f}_{1,n'}^{Y_1}((x_1 \oplus x'_1)|a_1| \dots |a_{n'}) \in Y_1$ and $\tilde{f}_{1,n'}^{Y'_1}((x_1 \oplus x'_1)|a_1| \dots |a_{n'}) \in Y'_1$ in (47) unchanged: then (46) follows and commutativity of (42) is proven.

- If $r \geq 3$ or $r' \geq 3$ in (43), one needs to further decompose the objects X'_{r-1} and $Y'_{r'-1}$ using the above techniques, i.e. introducing suitable exact triangles, arriving at the isomorphisms

$$(48) \quad \rho'_X : X \rightarrow X_1 \oplus X''_1 \oplus \dots \oplus X''_{r-1}, \quad \rho'_Y : Y \rightarrow Y_1 \oplus Y''_1 \oplus \dots \oplus Y''_{r'-1};$$

with $X_1, X''_1, \dots, X''_{r-1}, Y_1, Y''_1, \dots, Y''_{r'-1} \in \mathcal{S}_1$. We have reduced our problem to a finite direct sum of the \mathcal{S}_1 case. There is no substantial difference with the $r = 2, r' = 2$ case, both conceptually and computationally. We conclude that $\mathcal{G} \circ \mathcal{F} \simeq 1$ on $\mathbf{triang}_A^\infty(A)$.

On morphisms of objects in \mathcal{S}'_1 .

Let us consider \mathcal{S}'_1 and a strictly unital A_∞ -morphism $g : K\langle i \rangle[l] \rightarrow K\langle j \rangle[r]$ with Taylor components $\bar{g}_n : K\langle i \rangle[l+1] \otimes B[1]^{\otimes n} \rightarrow K\langle j \rangle[r+1]$, for $n \geq 0$. Once again, as $K\langle j \rangle[r+1]$ is concentrated in bidegree $(-r-1, -j)$ and g is of bidegree $(0, 0)$, then $\bar{g}_n = 0$ for $n \neq r-l+j-i$: there exists one and only one non trivial component \bar{g}_n . We check the commutativity of any diagram

$$(49) \quad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow \psi_X & & \downarrow \psi_Y \\ \mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y)) \end{array}$$

with X and Y in \mathcal{S}'_1 and ψ_X, ψ_Y natural quasi-isomorphisms in $\mathbf{D}^\infty(B)$. We need some preliminary results to prove this statement.

We recall that the A_∞ - B - A -bimodules $\underline{K}, \overline{K}$ are such that A_∞ quasi-isomorphisms of strictly unital A_∞ -bimodules $\nu_A : A \rightarrow \underline{K} \otimes_B \underline{K}$, and $\nu_B : B \rightarrow \overline{K} \otimes_A \overline{K}$ exist. Introducing the functor

$$(50) \quad \bar{\mathcal{G}} : \mathbf{D}^\infty(B) \rightarrow \mathbf{D}^\infty(A), M \mapsto \bar{\mathcal{G}}(M) := M \otimes_B \overline{K},$$

we prove that

- Lemma 14.**
- $\bar{\mathcal{G}}(M) \simeq \mathcal{G}(M)$ in $\mathbf{D}^\infty(A)$ ⁴, for any $M \in \mathbf{D}^\infty(B)$.
 - The functor $\bar{\mathcal{G}}$ is exact w.r.t. the triangulated structures on $\mathbf{D}^\infty(B)$ and $\mathbf{D}^\infty(A)$;
 - $\bar{\mathcal{G}}(M) \in \mathbf{triang}_A^\infty(A)$, for any object M in $\mathbf{triang}_B^\infty(K)$.

Proof. Let us consider the following diagram:

$$\begin{array}{ccccc}
 & & (M \otimes_B \bar{K}) \otimes_A (K \otimes_B \underline{K}) & & \\
 & \nearrow^{1 \otimes \nu_A} & & \nwarrow^{1 \otimes \nu_B \otimes 1} & \\
 & (M \otimes_B \bar{K}) \otimes_A A & & M \otimes_B (B \otimes_B \underline{K}) & \\
 \nearrow^{\Phi_{M \otimes_B \bar{K}}} & & & & \nwarrow^{1 \otimes \Phi_K} \\
 M \otimes_B \bar{K} = \bar{\mathcal{G}}(M) & & & & \mathcal{G}(M) = M \otimes_B \underline{K}
 \end{array}
 \tag{51}$$

where

$$\Phi_{M \otimes_B \bar{K}} : M \otimes_B \bar{K} \rightarrow (M \otimes_B \bar{K}) \otimes_A A$$

and similarly for $\Phi_{\underline{K}}$ are the maps described in lemma 3. All arrows are quasi-isomorphisms of strictly unital A_∞ - A - A -bimodules, i.e. homotopy equivalences. We denote by

$$\varphi_M : \bar{\mathcal{G}}(M) \rightarrow \mathcal{G}(M)$$

the above quasi-isomorphism in $\mathbf{D}^\infty(A)$, obtained by inverting the quasi-isomorphisms

$$\eta_{M,B} := 1 \otimes \nu_B \otimes 1,$$

and $1 \otimes \Phi_{\underline{K}}$; their inverses exist as quasi-isomorphisms in $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$ are homotopy equivalences. In other words,

$$\varphi_M = (1 \otimes \Phi_{\underline{K}})^{-1} \circ \eta_{M,B}^{-1} \circ (1 \otimes \nu_A) \circ (\Phi_{M \otimes_B \bar{K}}) := (\varphi_M^2)^{-1} \circ \varphi_M^1.$$

i.e. φ_M^1 is the composition on the l.h.s. of (51), while φ_M^2 is the one on the r.h.s. We also introduce the notation

$$T(M) = (M \otimes_B \bar{K}) \otimes_A (K \otimes_B \underline{K}) \cong M \otimes_B (\bar{K} \otimes_A K) \otimes_B \underline{K}.$$

By definition φ_M is not strict.

The first statement of the lemma follows by the very definition of φ_M .

The second statement is proved easily using the same techniques showing that \mathcal{G} is an exact functor w.r.t. the triangulated structures on $\mathbf{D}^\infty(B)$ and $\mathbf{D}^\infty(A)$. One the third statement; choosing $M = K$, then (51) implies $\bar{\mathcal{G}}(K) \simeq A$, or $\bar{\mathcal{G}}(X) \in \mathcal{S}_1 = \{A[i]\langle j \rangle, i, j \in \mathbb{Z}\}$, for any $X \in \mathcal{S}'_1 = \{K[i]\langle j \rangle, i, j \in \mathbb{Z}\}$. By definition of \mathcal{S}_r and \mathcal{S}'_r we have $\bar{\mathcal{G}}(X) \in \mathcal{S}_r$, for any $X \in \mathcal{S}'_r$. □

Corollary 10. $\mathcal{F} \circ \bar{\mathcal{G}} \simeq 1$ on $\mathbf{triang}_B^\infty(K)$.

Proof. Thanks to lemma 14, if $M \in \mathbf{triang}_B^\infty(K)$, then $\mathcal{G}(M) \simeq \bar{\mathcal{G}}(M)$ and $\mathcal{F}(\mathcal{G}(M)) \simeq \mathcal{F}(\bar{\mathcal{G}}(M))$, as \mathcal{F} sends quasi-isomorphisms to quasi-isomorphisms. It is easy to prove that $\mathcal{F} \circ \bar{\mathcal{G}} \simeq 1$ on $\mathbf{triang}_B^\infty(K)$ by following the subsection “Induction: \mathcal{S}_r ” above. Checking the commutativity of

$$\begin{array}{ccc}
 X & \xrightarrow{g} & Y \\
 \downarrow \psi_X & & \downarrow \psi_Y \\
 \mathcal{F}(\bar{\mathcal{G}}(X)) & \xrightarrow{\mathcal{F}(\bar{\mathcal{G}}(g))} & \mathcal{F}(\bar{\mathcal{G}}(Y))
 \end{array}$$

for any $X, Y \in \mathcal{S}'_1$ is immediate; note that ψ_X and ψ_Y are explicit; in fact

$$\psi_X : X \rightarrow \mathcal{F}(\bar{\mathcal{G}}(X)), \quad X \in \mathcal{S}'_1$$

is given by (up to suspensions and desuspensions)

$$\psi_X(x|b_1| \dots |b_n) = \sum_{n' \geq 0} ((x, b_1| \dots |b'_n| \mathcal{N}(\mathbf{R}_B(1)))|b_{n'+1}| \dots |b_n),$$

⁴More precisely, the quasi-isomorphisms are all of strictly unital A_∞ - A - A -bimodules.

where $\mathcal{N} : \underline{\text{End}}_A(K) \rightarrow \overline{K} \otimes_A K$ is s.t. $\mathcal{N}(\text{R}_B(1)) = \varphi \otimes 1$, with $\varphi : (A \otimes_A K)_0^0 \rightarrow K$, and $\varphi(1 \otimes 1) = 1$. This follows from the very definition of $\text{R}_B(1)$; in fact $\text{R}_B(1)(a_1 | \dots | a_l | 1) = 0$ for $l \geq 1$ and $\text{R}_B(1)(1) = 1 \cdot 1 = 1$. The relations

$$\mathcal{F}(\bar{\mathcal{G}}(g)) \circ \psi_X = \psi_Y \circ g,$$

i.e.

$$(55) \quad (g \otimes 1) \circ \psi_X = \psi_Y \circ g$$

follow, using (54). Then one moves to diagrams in which $X, Y \in \mathbf{triang}_B^\infty(K)$ and any morphism $g : X \rightarrow Y$ in $\mathbf{D}^\infty(B)$ appear; the proof of commutativity is done as in subsection “*Induction: \mathcal{S}_r* ”. We decompose objects in \mathcal{S}'_r , $r \geq 2$ into direct sums of objects in \mathcal{S}'_1 ; as the case for $r = 1$ is explicit, thanks to (54) and (55), then we can repeat *verbatim* the considerations in “*Induction: \mathcal{S}_r* ”, ending the proof of the equivalence $\mathcal{F} \circ \bar{\mathcal{G}} \simeq 1$ on $\mathbf{triang}_B^\infty(K)$. \square

We are left to prove

$$\mathcal{F} \circ \mathcal{G} \simeq 1$$

on $\mathbf{triang}_B^\infty(K)$ by induction on \mathcal{S}'_r , $r \geq 1$; we begin with the case $r = 1$. Let X and Y be in \mathcal{S}'_1 ; any diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow \rho_X & & \downarrow \rho_Y \\ \mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y)) \end{array}$$

can be decomposed into the subdiagrams

$$(56) \quad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow \psi_X & & \downarrow \psi_Y \\ \mathcal{F}(\bar{\mathcal{G}}(X)) & \xrightarrow{\mathcal{F}(\bar{\mathcal{G}}(g))} & \mathcal{F}(\bar{\mathcal{G}}(Y)) \\ \downarrow \mathcal{F}(\varphi_X) & & \downarrow \mathcal{F}(\varphi_Y) \\ \mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y)) \end{array}$$

where ψ_X, ψ_Y are given by (54), φ_X and φ_Y by (52) and $\rho_X = \mathcal{F}(\varphi_X) \circ \psi_X$, $\rho_Y = \mathcal{F}(\varphi_Y) \circ \psi_Y$ are quasi-isomorphisms.

We have already proved that the upper subdiagram in (56) commutes; the lower one commutes if we prove the commutativity of the diagram

$$(57) \quad \begin{array}{ccc} \bar{\mathcal{G}}(X) & \xrightarrow{\bar{\mathcal{G}}(g)} & \bar{\mathcal{G}}(Y) \\ \downarrow \varphi_X & & \downarrow \varphi_Y \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(Y) \end{array}$$

as \mathcal{F} is a functor. Once again, using the definition (52) of φ_X and φ_Y we decompose (57) into

$$\begin{array}{ccc} \bar{\mathcal{G}}(X) & \xrightarrow{\bar{\mathcal{G}}(g)} & \bar{\mathcal{G}}(Y) \\ (1 \otimes \nu_A) \circ (\Phi_{X \otimes_B \overline{K}}) \downarrow & & \downarrow (1 \otimes \nu_A) \circ (\Phi_{Y \otimes_B \overline{K}}) \\ T(X) & \xrightarrow{T(g)} & T(Y) \\ (1 \otimes \Phi_K)^{-1} \circ \eta_{X,B}^{-1} \downarrow & & \downarrow (1 \otimes \Phi_K)^{-1} \circ \eta_{Y,B}^{-1} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(Y) \end{array}$$

where the morphisms appear in the definition (52) and $T(X)$ (similarly for $T(Y)$) is defined in (53).

The map $T(g) : T(X) \rightarrow T(Y)$ is simply $T(g) = g \otimes 1$.

But

$$(58) \quad (1 \otimes \nu_A) \circ (\Phi_{Y \otimes_B \overline{K}}) \circ \bar{\mathcal{G}}(g) = T(g) \circ (1 \otimes \nu_A) \circ (\Phi_{X \otimes_B \overline{K}}),$$

as one can easily check just using the definitions of the morphisms; in fact the identity has to be verified on any string, say

$$(x, b_1 | \dots | b_q, \bar{\phi}) | a_1 | \dots | a_n \in (X \otimes_B \overline{K})[1] \otimes T(A[1]);$$

The l.h.s. of (58) is equal to (up to signs)

$$g((x, b_1 | \dots | b'_q), b_{q'+1} | \dots | b_q, \bar{\phi}) | a_1 | \dots | a_n, \nu_A(1),$$

as the morphism $g : X \rightarrow Y$ (X and Y are in \mathcal{S}'_1) has only one non trivial Taylor component $g_{q'}$, $q' \geq 0$, due to the bigrading on X and Y ; we already used this fact in the proof of thm. 7. The r.h.s. of (58) gives the same result, by definition of $T(g)$, which is clearly a strict A_∞ -morphism.

On the other hand

$$\mathcal{G}(g) \circ (1 \otimes \Phi_{\underline{K}})^{-1} \circ \eta_{X,B}^{-1} = (1 \otimes \Phi_{\underline{K}})^{-1} \circ \eta_{Y,B}^{-1} \circ T(g)$$

holds true if and only if

$$(59) \quad \eta_{Y,B} \circ (1 \otimes \Phi_{\underline{K}}) \circ \mathcal{G}(g) = T(g) \circ \eta_{X,B} \circ (1 \otimes \Phi_{\underline{K}});$$

(59) is easily verified, as we did for (58); so (56) commutes. Let us verify (59) explicitly, on any string

$$(x, b_1 | \dots | b_q, \phi) | a_1 | \dots | a_n \in (X \otimes_B \underline{K})[1] \otimes T(A[1]), \quad n \geq 0.$$

If $n \geq 1$, as all morphisms in (59) are strict, then (59) is trivially verified. Note that, by definition, $\Phi_{\underline{K}}$ “sees” the left B -module structure on \underline{K} , i.e. $\Phi_{\underline{K}} : \underline{K} \rightarrow B \otimes_B \underline{K}$. If $n = 0$, recalling that g has only one non trivial Taylor component, say $g_{\bar{q}}$, $\bar{q} \geq 0$, due to the bigradings on X and Y , we arrive at

$$\sum_{q'=\bar{q}+1}^q g((x, b_1 | \dots | b_{\bar{q}}, b_{\bar{q}+1} | \dots | b_{q'}, \nu_B(1), b_{q'+1} | \dots | b_q, \bar{\phi}),$$

for the l.h.s. of (59) (up to suspensions and desuspensions). We recall that $\nu_B : B \rightarrow \overline{K} \otimes_A K$ is strictly unital, so $\nu_B(b_1 | \dots | b_j | 1 | b_{j+1} | \dots | b_{j'}) = 0$ if $j' \geq 1$. The r.h.s. of (59) gives the same result, as $T(g) = g \otimes 1$.

The first step of the induction is proven. If $X \in \mathcal{S}_r$ and $Y \in \mathcal{S}_{r'}$, then we prove the commutativity of

$$(60) \quad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow \psi_X & & \downarrow \psi_Y \\ \mathcal{F}(\bar{\mathcal{G}}(X)) & \xrightarrow{\mathcal{F}(\bar{\mathcal{G}}(g))} & \mathcal{F}(\bar{\mathcal{G}}(Y)) \\ \downarrow \mathcal{F}(\varphi_X) & & \downarrow \mathcal{F}(\varphi_Y) \\ \mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y)) \end{array}$$

introducing the isomorphisms

$$\rho_X : X \rightarrow \bigoplus_{i=1}^r X_i, \quad \rho_Y : Y \rightarrow \bigoplus_{j=1}^{r'} Y_j$$

in $\mathbf{D}^\infty(B)$, for some $X_1, \dots, X_r, Y_1, \dots, Y_{r'}$ in \mathcal{S}_1 . The considerations that lead us to prove (44) hold here, with due changes; we are just considering finite direct sums of A_∞ -modules and homotopy equivalences.

Exchanging A and B ; i.e. using the A_∞ - B - A -bimodule (\tilde{K}, d_K) and the new functors $\mathcal{F}'' = \cdot \otimes_B \tilde{K}$ and $\mathcal{G}'' = \cdot \otimes_A \tilde{K}$ we can prove the equivalence of the triangulated categories $\mathbf{triang}_B^\infty(B)$ and $\mathbf{triang}_A^\infty(\tilde{K})$ with the same techniques introduced above.

B.0.41. On thick subcategories. The statement on the thick subcategories follows by additivity of \mathcal{F} and \mathcal{G} (\mathcal{F}'' and \mathcal{G}'' as well), w.r.t. the coproduct in $\mathbf{D}^\infty(A)$ and $\mathbf{D}^\infty(B)$, i.e. the direct sum of strictly unital A_∞ -modules.

More precisely, let $X \in \mathbf{thick}_A^\infty(A)$; there exists a $Z \in \mathbf{triang}_A^\infty(A)$ s.t.

$$Z \simeq X \oplus Y,$$

for some $Y \in \mathbf{D}^\infty(A)$. Let us call such isomorphism φ_X , i.e. $\varphi_X : Z \rightarrow X \oplus Y$. It follows that $\mathcal{F}(X) \in \mathbf{thick}_B^\infty(K)$, as \mathcal{F} is additive and preserves quasi-isomorphisms. For any morphism $f : X_1 \rightarrow X_2$, with $X_1, X_2 \in \mathbf{thick}_A^\infty(A)$, we want to prove that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ \mathcal{G}(\mathcal{F}(X_1)) & \xrightarrow{\gamma_1} & \mathcal{G}(\mathcal{F}(X_2)) \end{array}$$

commutes, for some isomorphisms $\psi_i : X_i \rightarrow \mathcal{G}(\mathcal{F}(X_i))$. All we need is to check the commutativity of the diagram

$$(61) \quad \begin{array}{ccccccc} & & X_1 & \xrightarrow{f} & X_2 & & \\ & & \downarrow i_1 & & \downarrow i_2 & & \\ Z_1 & \xrightarrow{\varphi_1} & X_1 \oplus Y_1 & \xrightarrow{i_2 \circ f \circ \pi_1} & X_2 \oplus Y_2 & \xrightarrow{\varphi_2^{-1}} & Z_2 \\ \downarrow \psi_{Z_1} & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \psi_{Z_2} \\ \mathcal{G}(\mathcal{F}(Z_1)) & \xrightarrow{\mathcal{G}(\mathcal{F}(\varphi_1))} & \mathcal{G}(\mathcal{F}(X_1 \oplus Y_1)) & \xrightarrow{\mathcal{G}(\mathcal{F}(i_2 \circ f \circ \pi_1))} & \mathcal{G}(\mathcal{F}(X_2 \oplus Y_2)) & \xrightarrow{\mathcal{G}(\mathcal{F}(\varphi_2^{-1}))} & \mathcal{G}(\mathcal{F}(Z_2)) \\ & & \downarrow \mathcal{G}(\mathcal{F}(\pi_1)) & & \downarrow \mathcal{G}(\mathcal{F}(\pi_2)) & & \\ & & \mathcal{G}(\mathcal{F}(X_1)) & \xrightarrow{\mathcal{G}(\mathcal{F}(f))} & \mathcal{G}(\mathcal{F}(Y_1)) & & \end{array}$$

with

$$\rho_1 := \mathcal{G}(\mathcal{F}(\varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1}$$

and similarly for ρ_2 . The isomorphisms ψ_{Z_i} do exist as $Z_i \in \mathbf{triang}_A^\infty(A)$. The maps $\pi_j : X_j \oplus Y_j \rightarrow X_j$ and $i_j : X_j \rightarrow X_j \oplus Y_j$ are morphisms in $\mathbf{D}^\infty(A)$. We want to prove that the central square in (61) commutes, i.e.

$$(62) \quad \rho_2 \circ (i_2 \circ f \circ \pi_1) = \mathcal{G}(\mathcal{F}(i_2 \circ f \circ \pi_1)) \circ \rho_1;$$

clearly

$$\begin{array}{ccc} Z_1 & \xrightarrow{\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1} & Z_2 \\ \downarrow \psi_{Z_1} & & \downarrow \psi_{Z_2} \\ \mathcal{G}(\mathcal{F}(Z_1)) & \xrightarrow{\mathcal{G}(\mathcal{F}(\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1))} & \mathcal{G}(\mathcal{F}(Z_2)) \end{array}$$

commutes, as $Z_i \in \mathbf{triang}_A^\infty(A)$; in other words

$$\begin{aligned} \psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1 &= \mathcal{G}(\mathcal{F}(\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \Leftrightarrow \\ \psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) &= \mathcal{G}(\mathcal{F}(\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1} \Leftrightarrow \\ \mathcal{G}(\mathcal{F}(\varphi_2) \circ \psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1)) &= \mathcal{G}(\mathcal{F}((i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1}, \end{aligned}$$

i.e. (62). The upper central and the lower central squares in (61) clearly commute; the morphisms

$$\psi_j : X_j \rightarrow \mathcal{G}(\mathcal{F}(X_j)), \quad \psi_j = \mathcal{G}(\mathcal{F}(\pi_j)) \circ \rho_j \circ i_j$$

are actually isomorphisms with inverses given by

$$\psi_j^{-1} : \mathcal{G}(\mathcal{F}(X_j)) \rightarrow X_j, \quad \psi_j^{-1} = \pi_j \circ \rho_j^{-1} \circ \mathcal{G}(\mathcal{F}(i_j));$$

this last statement is proved by writing explicitly ρ_j and recalling the decompositions (48), for any object in $\mathbf{triang}_A^\infty(A)$. As $\mathcal{G}(\mathcal{F}(\cdot))$ is of the form $\mathcal{G}(\mathcal{F}(g)) = g \otimes 1$, for any morphism g in $\mathbf{triang}_A^\infty(A)$, the claim follows. The thick subcategory $\mathbf{thick}_B^\infty(K)$ is studied analogously. \square

APPENDIX C. PROOF OF THM. 9

We show the proof of thm. 9 in some detail. such proof is analogous to the one of thm. 7, modulo technical issue due to the presence of “roofs”. Once again, all we need is to prove the commutativity of “easier” diagrams in which objects of the form $A_h[i]\langle j \rangle$ and $K_h[n]\langle m \rangle$ appear.

Proof. We study the exact⁵ functors \mathcal{F}_h and \mathcal{G}_h on $\mathbf{D}_{tf}^\infty(A_h)$ and $\mathbf{D}_{tf}^\infty(B_h)$ to prove that they induce an equivalence of triangulated categories between $\mathbf{triang}_{A_h}^\infty(\tilde{K}_h)$ and $\mathbf{triang}_{B_h}^\infty(B_h)$.

On objects in $\mathcal{S}_1, \mathcal{S}'_1$.

We introduce $\mathcal{S}_1 = \{A_h\langle i \rangle[n], i, n \in \mathbb{Z}\}$, and $\mathcal{S}'_1 = \{K_h\langle i \rangle[n], i, n \in \mathbb{Z}\}$. By definition, objects of \mathcal{S}_1 are (all isomorphism classes of the) objects $A_h\langle i \rangle[n]$ in $\mathbf{D}_{tf}^\infty(A_h)$ and similarly for \mathcal{S}'_1 . By definition of the functors $(\mathcal{F}_h, \mathcal{G}_h)$ and proposition 19 we have

$$\mathcal{G}_h(\mathcal{F}_h(A_h\langle i \rangle[n])) \simeq A_h\langle i \rangle[n]$$

⁵as usual, “exact” is w.r.t the triangulated structures on the derived categories

in $\mathbf{D}_{tf}^\infty(A_\hbar)$ and

$$\mathcal{F}_\hbar(\mathcal{G}_\hbar(K_\hbar\langle j\rangle[m])) \simeq K_\hbar\langle j\rangle[m]$$

in $\mathbf{D}_{tf}^\infty(B_\hbar)$, with $\mathcal{F}_\hbar(X) \in \mathcal{S}'_1$ and $\mathcal{G}_\hbar(Y) \in \mathcal{S}_1$, for every $X \in \mathcal{S}_1$, $Y \in \mathcal{S}'_1$.

C.0.42. *On commutative diagrams in the derived categories $\mathbf{D}_{tf}^\infty(A_\hbar)$ and $\mathbf{D}_{tf}^\infty(B_\hbar)$.* To prove the theorem, we need to consider the following general setting. Let X_\hbar, Y_\hbar be objects in $\mathbf{triang}_{A_\hbar}^\infty(A_\hbar)$ and let $f_\hbar : X_\hbar \rightarrow Y_\hbar$ be a morphism in $\mathbf{D}_{tf}^\infty(A_\hbar)$. We want to prove that there exist isomorphisms

$$\varphi_{X_\hbar} : X_\hbar \rightarrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(X_\hbar)), \quad \varphi_{Y_\hbar} : Y_\hbar \rightarrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(Y_\hbar))$$

in $\mathbf{D}_{tf}^\infty(A_\hbar)$, such that

$$(63) \quad \begin{array}{ccc} X_\hbar & \xrightarrow{f_\hbar} & Y_\hbar \\ \varphi_{X_\hbar} \downarrow & & \varphi_{Y_\hbar} \downarrow \\ \mathcal{G}_\hbar(\mathcal{F}_\hbar(X_\hbar)) & \xrightarrow{\mathcal{G}_\hbar(\mathcal{F}_\hbar(f_\hbar))} & \mathcal{G}_\hbar(\mathcal{F}_\hbar(Y_\hbar)) \end{array}$$

commutes in $\mathbf{D}_{tf}^\infty(A_\hbar)$.

Let us represent the morphism f_\hbar by the roof (s_\hbar, \bar{f}_\hbar) , i.e.

$$\begin{array}{ccc} & X'_\hbar & \\ s_\hbar \swarrow & & \searrow \bar{f}_\hbar \\ X_\hbar & & Y_\hbar \end{array}$$

for some X'_\hbar in $\mathbf{D}_{tf}^\infty(A_\hbar)$; by definition of $\mathbf{triang}_{A_\hbar}^\infty(A_\hbar)$ (property (SO)) we can infer that X'_\hbar is an object in $\mathbf{triang}_{A_\hbar}^\infty(A_\hbar)$: in fact s_\hbar is an isomorphism in $\mathbf{D}_{tf}^\infty(A_\hbar)$. Then the morphism $\mathcal{G}_\hbar(\mathcal{F}_\hbar(X_\hbar)) \rightarrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(Y_\hbar))$ is represented by the roof $(\mathcal{G}_\hbar(\mathcal{F}_\hbar(s_\hbar)), \mathcal{G}_\hbar(\mathcal{F}_\hbar(\bar{f}_\hbar)))$, i.e.

$$\begin{array}{ccc} & \mathcal{G}_\hbar(\mathcal{F}_\hbar(X'_\hbar)) & \\ \mathcal{G}_\hbar(\mathcal{F}_\hbar(s_\hbar)) \swarrow & & \searrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(\bar{f}_\hbar)) \\ \mathcal{G}_\hbar(\mathcal{F}_\hbar(X_\hbar)) & & \mathcal{G}_\hbar(\mathcal{F}_\hbar(Y_\hbar)) \end{array}$$

We are interested in proving also the commutativity of diagrams

$$(64) \quad \begin{array}{ccc} W_\hbar & \xrightarrow{g_\hbar} & Z_\hbar \\ \varphi_{W_\hbar} \downarrow & & \varphi_{Z_\hbar} \downarrow \\ \mathcal{F}_\hbar(\mathcal{G}_\hbar(W_\hbar)) & \xrightarrow{\mathcal{F}_\hbar(\mathcal{G}_\hbar(g_\hbar))} & \mathcal{F}_\hbar(\mathcal{G}_\hbar(Z_\hbar)) \end{array}$$

in $\mathbf{D}_{tf}^\infty(B_\hbar)$, with W_\hbar and Z_\hbar in $\mathbf{triang}_{B_\hbar}^\infty(K_\hbar)$, $\varphi_{W_\hbar}, \varphi_{Z_\hbar}$ isomorphisms in $\mathbf{D}_{tf}^\infty(B_\hbar)$ and morphisms g_\hbar represented by some roof, like the morphisms f_\hbar introduced above.

We need the following lemma, which reduces the problem of commutativity in the derived categories to the check of certain relations involving morphisms in the corresponding homotopy categories. We state the lemma in the case of diagrams of the form (63); the other case is analogous.

Lemma 15. *Let X_\hbar, Y_\hbar be objects in $\mathbf{triang}_{A_\hbar}^\infty(A_\hbar)$ and let us consider a diagram of the form (63), with $f_\hbar, \varphi_{X_\hbar}, \varphi_{Y_\hbar}$ and $\mathcal{G}_\hbar(\mathcal{F}_\hbar(f_\hbar))$ as above. If there exists a quasi-isomorphism*

$$\bar{\varphi}_{X'_\hbar} : X'_\hbar \rightarrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(X'_\hbar))$$

in $\mathcal{H}_\infty^{tf}(A_\hbar)$ s.t. for any morphism $g_\hbar : X'_\hbar \rightarrow Y_\hbar$ in $\mathcal{H}_\infty^{tf}(A_\hbar)$ the relation

$$(65) \quad \bar{\varphi}_{Y_\hbar} \circ g_\hbar = \bar{G}_\hbar(\bar{F}_\hbar(g_\hbar)) \circ \bar{\varphi}_{X'_\hbar}$$

holds true, then, representing the isomorphism

$$\varphi_{X'_\hbar} : X'_\hbar \rightarrow \mathcal{G}_\hbar(\mathcal{F}_\hbar(X'_\hbar)),$$

in $\mathbf{D}_{tf}^\infty(A_h)$ by the roof

$$\begin{array}{ccc} & X'_h & \\ 1_h \swarrow & & \searrow \bar{\varphi}_{X'_h} \\ X'_h & & \mathcal{G}_h(\mathcal{F}_h(X'_h)) \end{array}$$

the diagrams of the form (63) commute.

Proof. Commutativity of (63) is equivalent to

$$(66) \quad (1, \bar{\varphi}_{Y_h}) \circ (s_h, \bar{f}_h) = (\bar{G}_h(\bar{F}_h(s_h)), \bar{G}_h(\bar{F}_h(\bar{f}_h))) \circ (1, \bar{\varphi}_{X'_h})$$

in $\mathbf{D}_{tf}^\infty(A_h)$; the l.h.s. reads

$$(1, \bar{\varphi}_{Y_h}) \circ (s_h, \bar{f}_h) = (s_h \alpha_h, \bar{\varphi}_{Y_h} \beta_h),$$

with $\beta = \bar{f}_h \alpha_h$, for some roof

$$\begin{array}{ccc} & Z_h & \\ \alpha_h \swarrow & & \searrow \beta_h \\ X'_h & & Y_h \end{array}$$

Then

$$(1, \bar{\varphi}_{Y_h}) \circ (s_h, \bar{f}_h) = (s_h \alpha_h, \bar{\varphi}_{Y_h} \bar{f}_h \alpha_h) = (s_h, \bar{\varphi}_{Y_h} \bar{f}_h) = (s_h, \bar{G}_h(\bar{F}_h(\bar{f}_h)) \bar{\varphi}_{X'_h}),$$

where in the last equality we used (65) and the second equality holds true as α_h is a quasi-isomorphism⁶; but

$$(s_h, \bar{G}_h(\bar{F}_h(\bar{f}_h)) \bar{\varphi}_{X'_h}),$$

i.e. the roof

$$\begin{array}{ccc} & X'_h & \\ s_h \swarrow & & \searrow \bar{G}_h(\bar{F}_h(\bar{f}_h)) \bar{\varphi}_{X'_h} \\ X_h & & \mathcal{G}_h(\mathcal{F}_h(Y_h)) \end{array}$$

is equal to

$$\begin{array}{ccccc} & & X'_h & & \\ & & \swarrow s_h & \searrow \bar{\varphi}_{X'_h} & \\ & X_h & & \mathcal{G}_h(\mathcal{F}_h(X'_h)) & \\ & \swarrow 1_h & \searrow \bar{\varphi}_{X_h} \bar{G}_h(\bar{F}_h(s_h)) & \searrow \bar{G}_h(\bar{F}_h(\bar{f}_h)) & \\ X_h & & \mathcal{G}_h(\mathcal{F}_h(X_h)) & & \mathcal{G}_h(\mathcal{F}_h(Y_h)) \end{array}$$

i.e. the composition

$$(\bar{G}_h(\bar{F}_h(s_h)), \bar{G}_h(\bar{F}_h(\bar{f}_h))) \circ (1, \bar{\varphi}_{X'_h}),$$

which is the r.h.s. of (66), if and only if

$$\bar{\varphi}_{X_h} \circ s_h = \bar{G}_h(\bar{F}_h(s_h)) \circ \bar{\varphi}_{X'_h}.$$

This latter is nothing but (65) applied to the quasi-isomorphism s_h . □

In virtue of the above lemma, we prove that diagrams of the form (63) commute, for any X_h, Y_h in $\mathbf{triang}_{A_h}^\infty(A_h)$, by choosing a representative for the morphisms and checking the relations (66).

⁶We recall that equality “=” between roofs is, by definition, the equivalence relation between them.

C.0.43. On $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\mathbf{triang}_{A_h}^\infty(A_h)$. We begin by proving $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\mathbf{triang}_{A_h}^\infty(A_h)$ on any pair

$$X_h := A_h\langle i' \rangle[n'], \quad Y_h := A_h\langle j \rangle[m]$$

of objects in \mathcal{S}_1 and any morphism $f_h : A_h\langle i' \rangle[n'] \rightarrow A_h\langle j \rangle[m]$ in $\mathbf{D}_{tf}^\infty(A_h)$ represented by the roof

$$(67) \quad \begin{array}{ccc} & A_h\langle i \rangle[n] & \\ s_h \swarrow & & \searrow \bar{f}_h \\ A_h\langle i' \rangle[n'] & & A_h\langle j \rangle[m] \end{array}$$

where $i, i', n, n', j, m \in \mathbb{Z}$.

Let $\bar{f}_h^{(l),r} : (A\langle i \rangle[n])[1] \tilde{\otimes} A[1]^{\tilde{\otimes} r} \rightarrow (A\langle j \rangle[m])[1]$ be the r -th Taylor component of $f_h^{(l)}$ for any $l, r \geq 0$. A quick degree analysis (we recall that A_h is concentrated in cohomological degree 0) implies that $\bar{f}_h^{(l),r} \neq 0$ if and only if $r = m - n$, i.e. there exists one and only one non trivial Taylor component of $f_h^{(l)}$, if $m - n \geq 0$, for any $l \geq 0$. To prove the commutativity of the diagram

$$\begin{array}{ccc} A_h\langle i \rangle[n] & \xrightarrow{f_h} & A_h\langle j \rangle[m] \\ \downarrow \varphi_{A_h\langle i \rangle[n]} & & \downarrow \varphi_{A_h\langle j \rangle[m]} \\ \mathcal{G}_h(\mathcal{F}_h(A_h\langle i \rangle[n])) & \xrightarrow{\mathcal{G}_h(\mathcal{F}_h(f_h))} & \mathcal{G}_h(\mathcal{F}_h(A_h\langle j \rangle[m])) \end{array}$$

is sufficient, thanks to lemma 15 to prove

$$(68) \quad \bar{\varphi}_{A_h\langle j \rangle[m]} \circ \bar{f}_h = \mathcal{G}_h(\mathcal{F}_h(\bar{f}_h)) \circ \bar{\varphi}_{A_h\langle i \rangle[n]}$$

representing the isomorphisms

$$\varphi_{A_h\langle k \rangle[l]} : A_h\langle k \rangle[l] \rightarrow \mathcal{G}_h(\mathcal{F}_h(A_h\langle k \rangle[l]))$$

in $\mathbf{D}_{tf}^\infty(A_h)$ by

$$(69) \quad \begin{array}{ccc} & A_h\langle j \rangle[m] & \\ 1_h \swarrow & & \searrow \bar{\varphi}_{A_h\langle j \rangle[m]} \\ A_h\langle j \rangle[m] & & \mathcal{G}_h(\mathcal{F}_h(A_h\langle j \rangle[m])) \end{array}$$

for any $k, l \in \mathbb{Z}$. The quasi-isomorphisms $\bar{\varphi}_{A_h\langle j \rangle[m]}$ and $\bar{\varphi}_{A_h\langle i \rangle[n]}$ can be deduced by the diagram (40), with due changes. Up to suspensions and desuspensions w.r.t. both the cohomological and internal degree, we have

$$(70) \quad \bar{\varphi}_{A_h\langle i \rangle[n]} = (1 \tilde{\otimes} \mathcal{T}_h) \circ (1 \tilde{\otimes} L_{A_h}) \circ \Phi_{A_h},$$

where $\Phi_{A_h} : A_h \rightarrow A_h \tilde{\otimes}_{A_h} A_h$ and $\mathcal{T}_h : \underline{\text{End}}_{B_h}(K_h) \rightarrow K_h \tilde{\otimes}_{B_h} K_h$ is described in Corollary 5. To check (68) is immediate, once we recall that $\mathcal{G}_h(\mathcal{F}_h(\bar{f}_h)) = \bar{f}_h \tilde{\otimes} 1$.

We finish the proof of the equivalence $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\mathbf{triang}_{A_h}^\infty(A_h)$ considering the general case. Denoting by

$$\mathcal{S}_r = \underbrace{\mathcal{S}_1 \star \cdots \star \mathcal{S}_1}_{r\text{-times}},$$

the r^{th} extension of \mathcal{S}_1 for every $r \geq 1$, we note that $\mathcal{F}_h(X_h) \in \mathcal{S}'_r$ for every $X_h \in \mathcal{S}_r$ and $\mathcal{G}_h(Y_h) \in \mathcal{S}_{r'}$ for every $Y_h \in \mathcal{S}'_{r'}$, and $r, r' \geq 1$.

Let X_h and Y_h be objects in $\mathbf{triang}_{A_h}^\infty(A_h)$ and $f_h : X_h \rightarrow Y_h$ be a morphism in $\mathbf{D}_{tf}^\infty(A_h)$ represented by the roof

$$\begin{array}{ccc} & X'_h & \\ s_h \swarrow & & \searrow \bar{f}_h \\ X_h & & Y_h \end{array}$$

It follows that X'_h is an object in $\mathbf{triang}_{A_h}^\infty(A_h)$ as well. we show that the diagram

$$(71) \quad \begin{array}{ccc} X_h & \xrightarrow{f_h} & Y_h \\ \varphi_{X_h} \downarrow & & \downarrow \varphi_{Y_h} \\ \mathcal{G}_h(\mathcal{F}_h(X_h)) & \xrightarrow{\mathcal{G}_h(\mathcal{F}_h(f_h))} & \mathcal{G}_h(\mathcal{F}_h(Y_h)) \end{array}$$

commutes in $\mathbf{D}_{tf}^\infty(A_h)$. To do so, we introduce the roofs

$$\begin{array}{ccccc} & X_h & & Y_h & & \mathcal{G}_h(\mathcal{F}_h(X'_h)) \\ & \swarrow 1_h & \searrow \bar{\varphi}_{X_h} & \swarrow 1_h & \searrow \bar{\varphi}_{Y_h} & \swarrow \mathcal{G}_h(\mathcal{F}_h(s_h)) \searrow \mathcal{G}_h(\mathcal{F}_h(\bar{f}_h)) \\ X_h & & \mathcal{G}_h(\mathcal{F}_h(X_h)) & Y_h & & \mathcal{G}_h(\mathcal{F}_h(X_h)) & \mathcal{G}_h(\mathcal{F}_h(Y_h)), \end{array}$$

representing φ_{X_h} , φ_{Y_h} and $\mathcal{G}_h(\mathcal{F}_h(f_h))$, for some $\bar{\varphi}_{X_h}$ and $\bar{\varphi}_{Y_h}$ still to define (see below). On the other hand, $\mathcal{G}_h(\mathcal{F}_h(\bar{f}_h)) = \bar{f}_h \otimes 1$. By definition of the triangulated subcategory $\mathbf{triang}_{A_h}^\infty(A_h)$, there exist $r', r \geq 0$ s.t. $X'_h \in \mathcal{S}_{r'}$ and $Y_h \in \mathcal{S}_r$, i.e. there exist exact triangles

$$(72) \quad X_h^1 \rightarrow X'_h \rightarrow \bar{X}_h^{r'-1} \xrightarrow{g_h} X_h^1[1], \quad Y_h^1 \rightarrow Y_h \rightarrow \bar{Y}_h^{r-1} \xrightarrow{h_h} Y_h^1[1],$$

in $\mathbf{D}^\infty(A)$ for some morphisms g_h and h_h , with $X_h^1, Y_h^1 \in \mathcal{S}_1$ and $\bar{X}_h^{r'-1} \in \mathcal{S}_{r'-1}$, $\bar{Y}_h^{r-1} \in \mathcal{S}_{r-1}$. Let us focus on the first exact triangle in (72); for the second one the analysis is analogous. By definition of exact triangles in $\mathbf{D}_{tf}^\infty(A_h)$ (and $\mathbf{triang}_{A_h}^\infty(A_h)$), such exact triangle is isomorphic in $\mathbf{D}_{tf}^\infty(A_h)$ to a sequence of the form

$$(73) \quad W_h \xrightarrow{\alpha_h} Z_h \xrightarrow{\beta_h} R_h \xrightarrow{\gamma_h} W_h[1],$$

where (73) is the image under the canonical functor $\mathcal{Q}_{A_h} : \mathcal{H}_{tf}^\infty(A_h) \rightarrow \mathbf{D}^\infty(A_h)$ of the exact triangle $W_h \xrightarrow{\bar{\alpha}_h} Z_h \xrightarrow{\bar{\beta}_h} R_h \xrightarrow{\bar{\gamma}_h} W_h[1]$. In other words, the morphism α_h is represented by the roof

$$\begin{array}{ccc} & W_h & \\ 1_h \swarrow & & \searrow \bar{\alpha}_h \\ W_h & & Z_h; \end{array}$$

and similarly for β_h, γ_h .

But (73) is isomorphic in $\mathbf{D}_{tf}^\infty(A_h)$ to the exact triangle

$$(74) \quad W_h \xrightarrow{i_h} W_h \tilde{\oplus} R_h \xrightarrow{p_h} R_h \xrightarrow{\gamma_h} W_h[1], \quad d_{W_h \tilde{\oplus} R_h} = \begin{pmatrix} d_{W_h} & -\bar{\gamma}_h \\ 0 & d_{R_h} \end{pmatrix},$$

where i_h and p_h are represented by

$$\begin{array}{ccc} & W_h & \\ 1_h \swarrow & & \searrow \bar{i}_h \\ W_h & & W_h \tilde{\oplus} R_h, \end{array} \quad \begin{array}{ccc} & W_h \tilde{\oplus} R_h & \\ 1_h \swarrow & & \searrow \bar{p}_h \\ W_h \tilde{\oplus} R_h & & R_h, \end{array}$$

with canonical topological inclusion \bar{i}_h and topological projection \bar{p}_h . In summary, collecting the isomorphisms of the exact triangles so far, we arrive at the isomorphisms $X_h^1 \simeq W_h$ and $\bar{X}_h^{r'-1} \simeq R_h$ in $\mathbf{D}_{tf}^\infty(A_h)$, implying that $W_h \in \mathcal{S}_1$ and $R_h \in \mathcal{S}_{r'-1}$; the isomorphism

$$\rho_{X'_h} : (X'_h, d_{X'_h}) \rightarrow (W_h \tilde{\oplus} R_h, d_{W_h \tilde{\oplus} R_h}),$$

in $\mathbf{D}_{tf}^\infty(A_h)$ follows, as well. Repeating the same analysis for the exact triangle in which Y_h appears-see (72)-we get the isomorphism

$$\rho_{Y_h} : (Y_h, d_{Y_h}) \rightarrow (M_h \tilde{\oplus} N_h, d_{M_h \tilde{\oplus} N_h}),$$

in $\mathbf{D}_{tf}^\infty(A_h)$ for some $M_h \in \mathcal{S}_1$ and $N_h \in \mathcal{S}_{r-1}$.

In virtue of the above isomorphisms in $\mathbf{D}_{tf}^\infty(A_h)$, (71) commutes if and only if

$$(75) \quad \begin{array}{ccc} W_h \tilde{\oplus} R_h & \xrightarrow{\tilde{f}_h} & M_h \tilde{\oplus} N_h \\ \varphi_{W_h \tilde{\oplus} R_h} \downarrow & & \downarrow \varphi_{M_h \tilde{\oplus} N_h} \\ \mathcal{G}_h(\mathcal{F}_h(W_h \tilde{\oplus} R_h)) & \xrightarrow{\mathcal{G}_h(\mathcal{F}_h(\tilde{f}_h))} & \mathcal{G}_h(\mathcal{F}_h(M_h \tilde{\oplus} N_h)) \end{array}$$

does, where

$$(76) \quad \begin{aligned} \tilde{f}_h &= \rho_{Y_h} \circ f_h \circ \rho_{X_h}^{-1}, \\ \varphi_{W_h \tilde{\oplus} R_h} &= \rho_{\mathcal{G}_h(\mathcal{F}_h(X_h))} \circ \varphi_{X_h} \circ \rho_{X_h}^{-1}, \\ \varphi_{M_h \tilde{\oplus} N_h} &= \rho_{\mathcal{G}_h(\mathcal{F}_h(Y_h))} \circ \varphi_{Y_h} \circ \rho_{Y_h}^{-1}. \end{aligned}$$

In the sequel we will explicitly define $\varphi_{W_h \tilde{\oplus} R_h}$ and $\varphi_{M_h \tilde{\oplus} N_h}$; thanks to (76) φ_{X_h} and φ_{Y_h} will be explicit, as well. As we did in the proof of thm. 7 in the subsection “*Induction: \mathcal{S}_r* ”, we need to distinguish two cases: if $r' = r = 2$, i.e. $W_h, R_h, M_h, N_h \in \mathcal{S}_1$, we represent $\varphi_{W_h \tilde{\oplus} R_h}$ and $\varphi_{M_h \tilde{\oplus} N_h}$ by the roofs

$$\begin{array}{ccc} W_h \tilde{\oplus} R_h & & M_h \tilde{\oplus} N_h \\ \swarrow 1_h \quad \searrow \bar{\varphi}_{W_h \tilde{\oplus} R_h} & & \swarrow 1_h \quad \searrow \bar{\varphi}_{M_h \tilde{\oplus} N_h} \\ W_h \tilde{\oplus} R_h & \mathcal{G}_h(\mathcal{F}_h(W_h \tilde{\oplus} R_h)), M_h \tilde{\oplus} N_h & \mathcal{G}_h(\mathcal{F}_h(M_h \tilde{\oplus} N_h)), \end{array}$$

where $\bar{\varphi}_{W_h}, \dots, \bar{\varphi}_{N_h}$ are given by (70). Note that $\bar{\varphi}_{W_h} \tilde{\oplus} \bar{\varphi}_{R_h}$ and $\bar{\varphi}_{M_h} \tilde{\oplus} \bar{\varphi}_{N_h}$ are quasi-isomorphisms in $\mathcal{H}_\infty^{tf}(A_h)$ as, by definition, $\bar{\varphi}_{W_h}^{(0)}, \dots, \bar{\varphi}_{N_h}^{(0)}$ and are homotopy equivalences, i.e. isomorphisms, in $\mathbf{D}^\infty(A)$, as noted in the subsection “*Induction: \mathcal{S}_r* ” in the proof of thm. 7. Commutativity of (75) is easily proved: we are just following the lines of the proof in thm. 7, with due changes.

If $r' \geq 3$ or $r \geq 3$, we need to further decompose $\bar{X}_h^{r'-1}$ and \bar{Y}_h^{r-1} , repeating the above considerations, a finite number of times. The proof of commutativity of (75) is conceptually analogous to the one for the $r' = r = 2$ case; we are just considering a “trivially” quantized version of the computations which appear at the very end of the proof of thm. 7.

In summary, we have proven the equivalence $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\mathbf{triang}_{A_h}^\infty(A_h)$.

C.0.44. On $\mathcal{F}_h \circ \mathcal{G}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$. To prove $\mathcal{F}_h \circ \mathcal{G}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$ we begin by considering pair of objects in \mathcal{S}'_1 , following (once again!) the lines in the proof of thm. 7. We define the derived functor

$$\bar{\mathcal{G}}_h : \mathbf{D}_{tf}^\infty(B_h) \rightarrow \mathbf{D}_{tf}^\infty(A_h),$$

with $M_h \mapsto \bar{\mathcal{G}}_h(M_h) := M_h \tilde{\otimes}_{B_h} \bar{K}_h$

and we consider the following diagram:

$$\begin{array}{ccccc} & & M_h \tilde{\otimes}_{B_h} (\bar{K}_h \tilde{\otimes}_{A_h} K_h) \tilde{\otimes}_{B_h} K_h & & \\ & \nearrow 1 \otimes \nu_{A_h} & & \nwarrow 1 \otimes \nu_{B_h} \otimes 1 & \\ & & (M_h \tilde{\otimes}_{B_h} \bar{K}_h) \tilde{\otimes}_{A_h} A_h & & M_h \tilde{\otimes}_{B_h} (B_h \tilde{\otimes}_{B_h} K_h) \\ & \nearrow \Phi_{M_h \tilde{\otimes}_{B_h} \bar{K}_h} & & \nwarrow 1 \otimes \Phi_{K_h} & \\ \bar{\mathcal{G}}_h(M_h) = M_h \tilde{\otimes}_{B_h} \bar{K}_h & & & & M_h \tilde{\otimes}_{B_h} K_h = \mathcal{G}_h(M_h) \end{array}$$

where the quasi-isomorphisms $\Phi_{M_h \tilde{\otimes}_{B_h} \bar{K}_h}$ and Φ_{K_h} have been described in prop 15, while the quasi-isomorphisms ν_{A_h} and ν_{B_h} appear in Cor. 7. All arrows in the above diagram are quasi-isomorphisms of strictly unital topological A_∞ - A_h - A_h -bimodules. Introducing the notation

$$(77) \quad T_h(M_h) := M_h \tilde{\otimes}_{B_h} (\bar{K}_h \tilde{\otimes}_{A_h} K_h) \tilde{\otimes}_{B_h} K_h,$$

the quasi-isomorphism

$$\varphi_{M_h} : \bar{\mathcal{G}}_h(M_h) \rightarrow \mathcal{G}_h(M_h)$$

in $\mathbf{D}_{tf}^\infty(A_h)$ is the composition

$$(78) \quad \varphi_{M_h} = \alpha_{M_h} \circ \beta_{M_h},$$

choosing the roofs $(1_h, \bar{\alpha}_{M_h})$ and $(\bar{\beta}_{M_h}, 1_h)$ for α_{M_h} and β_{M_h} , where

$$\bar{\alpha}_{M_h} = (1 \tilde{\otimes} \nu_{A_h}) \circ \Phi_{M_h \tilde{\otimes}_{B_h} \bar{K}_h}, \quad \bar{\beta}_{M_h} = (1 \tilde{\otimes} \nu_{B_h} \tilde{\otimes} 1) \circ (1 \tilde{\otimes} \Phi_{K_h}).$$

\mathcal{F}_h and \mathcal{G}_h are exact w.r.t. the triangulated structures on $\mathbf{D}_{tf}^\infty(B_h)$ and $\mathbf{D}_{tf}^\infty(A_h)$; the statement for $\bar{\mathcal{G}}_h$ follows as well: the analysis is similar. Moreover $\bar{\mathcal{G}}_h(M_h) \in \mathbf{triang}_{A_h}^\infty(A_h)$, for any object M_h in $\mathbf{triang}_{B_h}^\infty(K_h)$; in fact

$$(79) \quad \bar{\mathcal{G}}_h(K_h) = K_h \tilde{\otimes}_{B_h} \bar{K}_h \simeq A_h$$

in $\mathbf{D}_{tf}^\infty(A_h)$ as it follows by considering the quasi-isomorphism φ_{K_h} and using $K_h \tilde{\otimes}_{B_h} K_h \simeq A_h$; the statement for any object in \mathcal{S}'_1 easily follows; for $r \geq 2$ we just need to look at exact triangles. We continue with the equivalence $\mathcal{F}_h \circ \bar{\mathcal{G}}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$; its proof is done by decomposing objects in \mathcal{S}'_r , $r \geq 2$, into finite direct sums of objects

in \mathcal{S}'_1 , as we did in the preceding subsection. We need to prove the step $r = 1$ explicitly. Let $K_h\langle i'\rangle[n']$, $K_h\langle j\rangle[r]$ be two objects in \mathcal{S}'_1 and let $g_h : K_h\langle i'\rangle[n'] \rightarrow K_h\langle j\rangle[r]$ be a morphism in $\mathbf{D}^\infty(B_h)$ represented by the roof

$$(80) \quad \begin{array}{ccc} & K_h\langle i\rangle[l] & \\ s_h \swarrow & & \searrow \bar{g}_h \\ K_h\langle i'\rangle[n'] & & K_h\langle j\rangle[r] \end{array}$$

where $i, i', l, n', j, r \in \mathbb{Z}$. By degree reasons \bar{g}_h has a unique non trivial Taylor component $\bar{g}_h^{(r)}$, for some $r \geq 0$. The commutativity of the diagram

$$\begin{array}{ccc} K_h\langle i'\rangle[n'] & \xrightarrow{g_h} & K_h\langle j\rangle[r] \\ \downarrow \psi_{K_h\langle i'\rangle[n']} & & \downarrow \psi_{K_h\langle j\rangle[r]} \\ \mathcal{F}_h(\mathcal{G}_h(K_h\langle i'\rangle[n'])) & \xrightarrow{\mathcal{F}_h(\bar{\mathcal{G}}_h(g_h))} & \mathcal{F}_h(\bar{\mathcal{G}}_h(K_h\langle j\rangle[r])) \end{array}$$

is proven once we show that

$$(81) \quad \begin{array}{ccc} K_h\langle i\rangle[l] & \xrightarrow{\bar{g}_h} & K_h\langle j\rangle[r] \\ \downarrow \mathcal{Z}_1^h & & \downarrow \mathcal{Z}_2^h \\ (K_h \tilde{\otimes}_{B_h} B_h)\langle i\rangle[l] & & (K_h \tilde{\otimes}_{B_h} B_h)\langle j\rangle[r] \\ \downarrow \mathcal{Z}_3^h & & \downarrow \mathcal{Z}_4^h \\ (K_h \tilde{\otimes}_{B_h} \underline{\text{End}}_{A_h}(K_h)^{op})\langle i\rangle[l] & & (K_h \tilde{\otimes}_{B_h} \underline{\text{End}}_{A_h}(K_h)^{op})\langle j\rangle[r] \\ \downarrow \mathcal{Z}_5^h & & \downarrow \mathcal{Z}_6^h \\ (K_h \tilde{\otimes}_{B_h} (\bar{K}_h \tilde{\otimes}_{A_h} K_h))\langle i\rangle[l] & \xrightarrow{\mathcal{F}_h(\bar{\mathcal{G}}_h(\bar{g}_h))} & (K_h \tilde{\otimes}_{B_h} (\bar{K}_h \tilde{\otimes}_{A_h} K_h))\langle j\rangle[r] \end{array}$$

commutes, where the morphisms \mathcal{Z}_i^h are those appearing in the proof of thm. 7, section 7, with due changes. Considering (81), given any $X_h \in \mathcal{S}'_1$, we denote by $\bar{\psi}_{X_h} : X_h \rightarrow \mathcal{F}_h(\bar{\mathcal{G}}_h(X_h))$ the composition

$$(82) \quad \bar{\psi}_{X_h} = \mathcal{Z}_5^h \circ \mathcal{Z}_3^h \circ \mathcal{Z}_1^h = (1 \tilde{\otimes} \mathcal{R}_h) \circ (1 \tilde{\otimes} R_{B_h}) \circ (\Phi_{K_h}),$$

where Φ_{K_h} is described in prop 15, R_{B_h} is the quantized derived right action and \mathcal{R}_h appears in cor 6. $\bar{\psi}_{X_h}$ is a quasi-isomorphism in $\mathcal{H}_{tf}^\infty(A_h)$ as $\bar{\psi}_{X_h}^{(0)}$ is a quasi-isomorphism, i.e. a homotopy equivalence in $\mathbf{D}^\infty(A)$.

Using the decomposition of objects in \mathcal{S}'_r into finite direct sums of objects in \mathcal{S}'_1 we finish the proof of the equivalence $\mathcal{F}_h \circ \bar{\mathcal{G}}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$. We can repeat the analysis of the previous subsection almost *verbatim*.

The equivalence $\mathcal{F}_h \circ \bar{\mathcal{G}}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$ is proved following the same strategy that lead us to $\mathcal{F}_h \circ \bar{\mathcal{G}}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$: all we need is to consider the case $r = 1$ using $\mathcal{F}_h \circ \bar{\mathcal{G}}_h \simeq 1$ on $\mathbf{triang}_{B_h}^\infty(K_h)$. All computations and decompositions that appear in the proof of thm. 7 can be repeated here, with due changes.

C.0.45. *Last part of the proof.* Exchanging A_h and B_h ; i.e. using the topological A_∞ - B_h - A_h -bimodule (\tilde{K}_h, d_{K_h}) and the new functors $\mathcal{F}_h'' = \cdot \tilde{\otimes}_{B_h} \tilde{K}_h$ and $\mathcal{G}_h'' = \cdot \tilde{\otimes}_{A_h} \tilde{K}_h$ we can prove the equivalence of the triangulated categories $\mathbf{triang}_{B_h}^\infty(B_h)$ and $\mathbf{triang}_{A_h}^\infty(\tilde{K}_h)$ with the same techniques introduced above.

The statement on the thick subcategories follows by additivity of \mathcal{F}_h and \mathcal{G}_h (\mathcal{F}_h'' and \mathcal{G}_h'' as well), w.r.t. the coproduct in $\mathbf{D}_{tf}^\infty(A_h)$ and $\mathbf{D}_{tf}^\infty(B_h)$, i.e. the direct sum of strictly unital topological A_∞ -modules. \square

APPENDIX D. ON TRIANGULATED CATEGORIES

In this section we collect some known facts on triangulated categories and thickness. We follow the expositions in [20] and [16]. Let us consider the pair (\mathcal{T}, Σ) , where \mathcal{T} is an additive category and $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$, $\Sigma(X) := \Sigma X$ an additive autoequivalence.

Definition 34. A triangle in \mathcal{T} is a triple (α, β, γ) of morphisms in \mathcal{T}

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma Z,$$

and a morphism bewteen two triangles (α, β, γ) , $(\alpha', \beta', \gamma')$ is a triple $(\varphi_1, \varphi_2, \varphi_3)$ of morphisms in \mathcal{T} s.t. the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \Sigma(\varphi_1) \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

commutes.

Definition 35. The category \mathcal{T} is said to be triangulated if it is equipped with a class of distinguished triangles, called the exact triangles, satisfying the following axioms.

- (T1) A triangle isomorphic to an exact triangle is exact. For any object X , the triangle $0 \rightarrow X \xrightarrow{1} X \rightarrow 0$ is exact. Any morphism $\alpha : X \rightarrow Y$ in \mathcal{T} can be completed to an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$.
- (T2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.
- (T3) Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of morphisms φ_1 and φ_2 satisfying $\varphi_2 \circ \alpha = \alpha' \circ \varphi_1$ can be completed (not necessarily uniquely) to a morphism of triangles $(\varphi_1, \varphi_2, \varphi_3)$:

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \Sigma(\varphi_1) \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

- (T4)⁷ Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ \parallel & & \downarrow \beta_1 & & \downarrow \delta_1 & & \parallel \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow \beta_2 & & \downarrow \delta_2 & & \downarrow \Sigma\alpha_1 \\ & & W & \xlongequal{\quad} & W & \xrightarrow{\beta_3} & \Sigma Y \\ & & \downarrow \beta_3 & & \downarrow \delta_3 & & \\ & & \Sigma Y & \xrightarrow{\Sigma(\alpha_2)} & \Sigma U & & \end{array}$$

commutative.

If the category \mathcal{T} satisfies only the axioms (T1)-(T2)-(T3), then it is said to be a pre-triangulated category.

Definition 36. Let \mathcal{T} be a pre-triangulated category and \mathbf{Ab} be the category of abelian groups. A functor $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{A}$, with \mathcal{A} abelian, is said to be cohomological if it sends each exact triangle in \mathcal{T} to an exact sequence in \mathcal{A} .

Lemma 16. For each $\in \mathcal{T}$, the representable functors

$$\mathrm{Hom}_{\mathcal{T}}(X, \cdot) : \mathcal{T} \rightarrow \mathbf{Ab}, \quad \mathrm{Hom}_{\mathcal{T}}(\cdot, X) : \mathcal{T}^{op} \rightarrow \mathbf{Ab},$$

are cohomological.

From the above lemma it follows

Lemma 17. Let $(\varphi_1, \varphi_2, \varphi_3)$ be a morphism between exact triangles in \mathcal{T} . If two maps in $\{\varphi_1, \varphi_2, \varphi_3\}$ are isomorphisms, then also the third.

Definition 37. Let (\mathcal{T}, Σ_1) and (\mathcal{U}, Σ_2) be triangulated categories. An exact functor $\mathcal{T} \rightarrow \mathcal{U}$ is a pair (\mathcal{F}, η) consisting of a functor $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{U}$ and a natural isomorphism $\eta : \mathcal{F} \circ \Sigma_1 \rightarrow \Sigma_2 \circ \mathcal{F}$ s.t., for every exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma_1 X$ in \mathcal{T} , the triangle

$$\mathcal{F}(X) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(Z) \xrightarrow{\eta \circ \mathcal{F}(\gamma)} \Sigma_2 \mathcal{F}(X)$$

is exact in \mathcal{U} .

⁷this is the celebrated octahedral axiom.

The following basic example is well studied in [11].

Example 2. Let (\mathcal{T}, Σ) be a triangulated category. The autoequivalence $(\Sigma, -1)$ is an exact functor w.r.t. the triangulated structure on \mathcal{T} .

D.0.46. On triangulated subcategories. Let (\mathcal{T}, Σ) be a triangulated category.

Definition 38. A non-empty full additive subcategory \mathcal{C} is a triangulated subcategory if

- (S0) \mathcal{S} is strict; any object isomorphic to an object in \mathcal{S} belongs to \mathcal{S} .
- (S1) $\Sigma^n X \in \mathcal{C}$ for all $X \in \mathcal{C}$ and $n \in \mathbb{Z}$.
- (S2) Let $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ be any exact triangle in \mathcal{T} . If any two objects from $\{X, Y, Z\}$ belong to \mathcal{C} , so also the third.

A triangulated subcategory \mathcal{C} inherits a canonical triangulated structure from \mathcal{T} . Let \mathcal{U} and \mathcal{V} be classes whose objects are (isomorphism classes of) objects in \mathcal{T} , where \mathcal{T} is triangulated. The class $\mathcal{U} \star \mathcal{V}$ is defined as follows:

$$\mathcal{U} \star \mathcal{V} := \{X \in \mathcal{T} : U \rightarrow X \rightarrow V \rightarrow \Sigma U \text{ exact triangle in } \mathcal{T}, U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The composition \star is associative by the octahedral axiom (T4). The following notation

$$\mathcal{S}_r = \underbrace{\mathcal{S}_1 \star \mathcal{S}_1 \star \cdots \star \mathcal{S}_1}_{r\text{-times}}$$

is unambiguous for $r \geq 1$, for any class \mathcal{S}_1 of objects in \mathcal{T} . The objects in \mathcal{S}_r are called the extensions of length r of objects of \mathcal{S}_1 . If \mathcal{T} is a triangulated category and M is an object in \mathcal{T} , the full triangulated subcategory generated by M consists of all objects belonging to \mathcal{S}_r ($r \geq 1$, as above) with $\mathcal{S}_1 = \{M[i], i \in \mathbb{Z}\}$ (in \mathcal{S}_1 we consider equivalence classes of isomorphic objects). Such triangulated subcategory is the smallest full triangulated subcategory in \mathcal{T} containing M .

Its thickening is the full triangulated subcategory of \mathcal{T} consisting of all objects X in \mathcal{T} , s.t. there exist an object Z in the triangulated subcategory generated by M with $Z \simeq X \oplus Y$. The thickening is closed under direct summands; actually it is the smallest full triangulated subcategory in \mathcal{T} containing the triangulated subcategory generated by M and closed under direct summands.

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